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THE EVALUATION OF WELL-BEING IN THE FUNCTIONING-CAPABILITY SPACE: A NEW CRITERION BASED ON REFINED FUNCTIONING

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The evaluation of well-being in the functioning-capability space: a new criterion based on refined functioning

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Abstract

It is generally agreed that the identification of income wealth does not provide all the relevant information needed to evaluate the quality of life: income is but one component of overall individual wellbeing. Latest economic contributions focus on the multidimensional nature of well-being taking into account a number of life dimensions further than income (e.g. health conditions, educational attainments). Moreover, as recently argued, freedom of choice plays a relevant role in the definition of quality of life. Which criterion could be the most appropriate to allow for the multidimensional nature of well-being? Can freedom of choice be included in such a comprehensive evaluation? This paper concerns a re-examination of the notion of well-being within the functioning-capability approach proposed by A.K. Sen. Following Sen's framework we re-define the value of achieved functionings in a way that takes note of alternative opportunities. We make operative a freedom-of-choice based refinement procedure by partitioning the population into different groups, homogeneous in some discriminating objective characteristic (e.g. age, sex, location). Our aim is to test whether the fact of showing a certain attribute poses an objective limit to a person's opportunity to reach or exceed a given value of a functioning.

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1 Introduction

The evaluation of well-being represents one of the most relevant issues in welfare economics: the importance of the definition and evaluation of standard of living lies in the fact that the chosen indicators will be used by policy makers in the actual design of redistributive policies.

To focus the problem, let's consider three persons. Person A lives in a very rich family but she has a serious handicap, and she has no alternatives but staying home: she cannot work and cultivate social relations. Person B is poor in income, but he enjoys a very good health status: he is a Ph.D. student in philosophy and he loves travelling. Finally, person C is a migrant, she has not high school level and she belongs to the working class. Which of the three persons has the higher well-being? Which of them deserve the highest support from the government?

According to traditional welfarist approach, which evaluates different social states in terms of individual utility generated by the consumption of commodities, and empirically measures welfare in terms of income or consumption (considered the best proxy of utility), person A enjoys the highest level of well-being, followed by C and B. Is this ranking correct? Is the traditional evaluation of well-being adequate?

Recently, an increasing literature has agreed the traditional approach is inadequate for an accurate evaluation of living standard. Scholars (see, among others, Barberà, Bossert and Pattanaik, 2003; Bossert, Pattanaik and Xu, 1994; Dworkin, 1981; Gravel, 1994; Ok, 1997; Pattanaik and Xu, 1990, 1998, 2000; Sen 1991, 1992; Sugden, 1998; Xu, 2002, 2004) argue for a more comprehensive interpretation of well-being, by focusing on two basic ideas: the first idea is that well-being represents a multidimensional notion, which only partially depends to economic wealth, and which is linked to dimensions like health condition, education, safety and self fulfillment. The second -perhaps more crucial- idea is the individual freedom of choice is relevant in determining the level of well-being.

Within this literature Sen (1985, 1987, 1992) proposed a functioning-capability approach, with the capability space being a particular space of individual opportunities and alternative achievements, among which a person can choose.

The main focus of this paper is to assess individual well-being in a way that takes note of its multidimensional nature and the role of freedom of choice. In the first section we present the functioning-capability approach proposed by professor A. K. Sen; in section two we provide some consideration about the opportunity to use Sen's view to face the issue of well-being assessment focusing on the notion of freedom of choice; section three concentrates on a set of criteria which incorporate an element of freedom of choice in the evaluation of living standard; in section four we present examples also based on empirical data. We conclude the paper motivating our suggestion for a further specification for one of the criteria presented in section 3. Under suitable properties our criterion will allow to derive robust individual comparisons in terms of the freedom of choice based refined functioning.

1.1 The capability approach

Since the '70s Sen launched a critique against the welfarism as a normative theory, and proposed a new framework the assessment of well-being, which he called capability approach: the capability approach stresses on the role of individual freedom of choice and interpersonal differences in the definition of human well-being.

Sen (1980, 1985, 1992) points out that there are dimensions of welfare which are not completely captured by either the income earned or the available resources. Income is necessary to buy commodities, which in turn help people to satisfy material needs; however, there exist elements of life (such as health, culture or self-respect) which do not depend upon material wealth. Thus, well-being is fundamentally a multidimensional and complex notion. Sen's approach is able to account for the multidimensional nature of well-being as it characterizes individual well-being in terms of what a person is actually able to do or to be, i.e. in terms of his/her functionings: a functioning is an achievement of a person, what the person succeeds in doing with the commodities and characteristics at her command (see Sen, 1985). Thus, the functioning vector represents the overall "state of being and doing" of a person.[1]

Related to that of functionings, Sen defines the concept of capability set. The capability set represents a person's opportunities to achieve well-being: it reflects the alternative combinations of functionings that a person may achieve, given her personal characteristics, the environmental circumstances, and her preferences. Thus the capability set incorporates the freedom of choice of agents[2]. An illustrative example may be useful to understand the notion of capability: suppose that a person can choose between different functioning vectors *a, b, c* and *d*, and that she chooses *a*. Now consider the case that vectors *b, c* and *d* become unavailable to the person, while *a* remains the only accessible alternative: at first glance, it seems that the agent's living standard remains unchanged as she would choose the vector *a* anyway. However, we must agree that the person's well-being has been reduced due to the limitation of her freedom of choice.

[1] Functionings include elementary things like being alive, being well-nourished or sheltered, being in good health. Also, there are more complex functionings such as having self-respect, cultivating intellectual attitudes, appearing in public without shame etc. The former are referred to as basic functionings, and they assume a key role especially in less developed societies in which a number of persons must face starvation and health disease; latter forms of functionings become fundamental in the richest societies, in which it is presumed that the majority of people has means to escape basic needs. Functionings must be distinguished from commodities: the latter are objects which persons might buy and use, while each functioning represents an aspect of living itself (see Sugden, 1993). Functionings are also to be distinguished from utilities: according to Sen utility takes the form of the subjective happiness resulting from the achievement of a functioning.

[2] In this paper the concept of freedom of choice must be interpreted in a positive rather than negative sense, "that is, in terms of 'freedom to' rather than 'freedom from"' (see Sugden, 1993). Sen argues that positive freedom of choice is a good on itself, has an intrinsic value: being free to choose how to live one's own life is one of the effective dimensions of well-being. At the same time, Sen points out that freedom of choice is not merely a functioning. Rather, it must be identified with the capability set.

Formally, consider x_i is a vector of commodities possessed by a person i, $c(.)$ is the function which converts commodities vector in a vector of characteristics of those commodities. Also, consider $f_i(.)$ being a personal utilization function, which transforms the characteristic vector $c(x_i)$ in a functioning vector b_i, where $b_i = f_i(c(x_i))$.Function $f_i(.)$ reflects the pattern of use of commodities that person i (and only person i) can actually make, given her personal abilities and the environmental circumstances. The way the commodity vector can be converted in a functioning vector, then, will depend on the own person characteristics.

Given this characterization of individual standard of living, the main problem now becomes how to assess well-being in the capability space.

2 Use the capability approach to evaluate individual well-being

In the first section we have briefly illustrated Sen's approach: now we attempt to motivate the opportunity to use Sen's framework for actual assessment of the living standard: how do we arrive to a measure of overall well-being which incorporate the notion of freedom of choice? How do we measure the freedom of choice?

In fact the multidimensional nature of well-being and the complex structure of the capability approach make its operationalization a difficult task. Such operationalization for practical assessment of human well being requires the dealing with a number of issues, and above, it is related with the choice of an adequate methodology for evaluating functioning indicators.

As we have seen in Section 1, the b_i vector represents the overall "*state of being and doing*" of person i: Sen points out the possibility to evaluate b_i by using an objective evaluation function $v(.)$. This objective evaluation function may represent the starting point for well-being assessment. However, function $v(.)$ is restricted to the evaluation of functionings, while it ignores a consideration of individuals' capability set as it does not take into account the individual's freedom of choice.[3]

In order to incorporate the freedom of choice in the analysis of individual well-being Sen provides a number of suggestions. Among them he introduces the notion of refined functioning: where the value of the achieved functionings evaluated according to $v(.)$ are refined by taking into account an element of freedom of choice. Complementing the value of $v(.)$ with the "freedom factor" will lead to a more accurate assessment of living standard.

[3] The choice between functioning vector and capability set is a fundamental one in the analysis and empirical evaluation of personal well-being. On the one hand, functionings are more closely related to the actual living standards of people: thus, we might argue that the assessment of the value of functioning vector is sufficient to reach an adequate evaluation of well-being. On the other hand, Sen considers the capability space the more adequate sphere for evaluating the level of well-being, as it embodies the freedom of individuals: by choosing the capability set as the relevant space for the analysis of well-being, we may assess the person's standard of living under different situation, and arrive to a more appropriate valuation of well-being.

According to Sen a *refined* functioning takes the form $(b_i|S)$ of "*reach the functioning b_i given the choosing set S*". This criterion tries to relate the value of functioning b_i with a choosing set S, that provides a metric of freedom of choice.

In the next section we will try to provide a characterization of an objective evaluation function of capability by applying the concept of refined functioning as described by Sen.

3 A freedom-of-choice based refinement function

The possibility to evaluate well-being which incorporates an element of freedom of choice is related to the definition of a functional representation of individual refined functionings. Thus the objective of this section is to define a *freedom-of-choice based refinement function.*

Different paths can be followed to accomplish the task: we follow the axiomatic approach proposed by recent literature which suggests to incorporate the notion of freedom of choice in the analysis of individual well-being by introducing the notion of opportunity set (see Peragine, 1998). Opportunity set is the set of all alternatives available to individual, from which she can choose the outcome she prefers (see Pattanaik and Xu 1990, 1998, Bossert et al. 1997, Kranich 1996, 1997).

If we combine the axiomatic methodology proposed by the literature on *opportunity sets* and the notional framework of the *capability approach,* we may arrive to characterize a *freedom of choice based refined functioning.* As a by-product of the analysis the methodology we suggest can be considered a possible solution to the problem of *operationalize the notion of capability set* of the individuals.

The formal framework employed here refers to the notation proposed in Sen (1985), but it can be also related to the literature on opportunity set ranking.

Suppose there is a non-empty universal set of functionings P, finite and with cardinality $|P|$. These functionings describe all possible states of being and doing available to individuals belonging to a community, so that P constitutes the set of fundamental components of well-being. Population is composed of n individuals, where n is a finite number. Each person i $(i = 1, 2, ..., n)$ has a non-empty and finite subset of P, which we call the capability set Q_i. Within the capability set each person may achieve a functioning vector, which represents her overall state of being and doing, embodying all the relevant dimensions of her life. The realization of functioning by the individual i depends on two main aspects:

1. her preferences;
2. and her freedom of choice.

Obviously we cannot provide an objective metric of preferences: yet, we are able to observe the concrete functioning achieved by each individual, and compare it with that of other individuals. Each functioning is determined by an indicator which can assume dichotomous or ordinal values. Given the observed indicator, it is possible to design the objective evaluative function $v(.)$ introduced in Section 1.

Once we have an opportune evaluation $v(.)$ of observed functioning, we wish to evaluate the freedom of choice of individual i by taking note of her alternative opportunities. But, how can we proceed to take note of alternatives? Theoretically, it would be necessary to compare the actual state of each individual with all alternative states the same individual could have reached given her personal characteristics, the environmental circumstances and the available commodities (that is to asses the value of capability set Q_i). However, *at a practical level*, this is unfeasible as we do not know the potential and unrealized alternative opportunities of agent i: yet, what we can do is to compare the actual state of each individual with that of individuals who share with her one of more personal objective characteristics (e.g. the age, the social background, the household composition), and then to weight the value of functioning vector according to the degree of freedom of choice correlated with the share of that characteristic. The intuitive hypothesis is that there are some objective characteristics which represent "factors of freedom", as they affect the capacity of individuals of exercising the freedom of choice; thus, we suppose groups have a constitutive importance in the definition of individuals' capabilities. The procedure considers a society and establishes an ordering between different groups in terms of freedom of choice: by choosing one or more discriminating characteristics (e.g. the gender and/or the class of age), we may group individuals in K different types according to that characteristic (if the characteristic is the gender, k will assumes the alternative types "male" and "female"). Each group is composed of n^k individuals. We call $S_k \in P$ the choosing set, including all individuals sharing the characteristic k: the fact that a person belongs to a certain subgroup may or may not limit her capacity to exercise the freedom of choice. The choosing set S_k represents a target to estimate the freedom of choice related to the possession of an objective characteristic, exogenous from individual's preferences: the higher the choosing space S_k, the larger the number of valuable alternatives among which the individual can choose. On the other hand, the actual realization in terms of functioning represents the preferences of individuals: given a (more or less extended) number of alternatives, the agent will choose the state of being or doing he/she prefers.

First we focus on one specific functioning, i.e. one specific aspect of overall well-being: our objective is to evaluate the refined functioning b_i^k achieved by individual.

Let defined the *refined functioning* of individual i $b_i^k = (b_i|S_k)$, with $b_i^k \in S_k$: the value of this functioning is able to take note of the 'counterfactual' opportunities of individual (see Gravel, 1994).

Starting from the objective evaluation function $v(b_i)$ and taking note of Sen's suggestion about refined functioning, we propose a new evaluation function $\omega(b_i|S_k)$.Each non-empty and finite set S_k can be identified with the n^k-dimensional vector (or the set) of its elements. Thus, it is possible to represent function $\omega(.)$ as $w(b_i; \mathbf{b}^k)$, where b_i represents the functioning reached by individual i, and $\mathbf{b}^k$ represents the vector of functionings achieved by individuals (included individual i) belonging to the choosing set S_k. The set of all functioning vectors can be denoted by $\mathbf{B}$, with $\mathbf{b}^k \in \mathbf{B}$, the subset of all vectors with size n is denoted by $\mathbf{B}_n$.

We will consider levels of functioning achieved b_i for agent i where given a fixed level H of realizations the set of possible realizations is $N_H := \{1, 2, 3...H\}$ where $H \geq 2$. Similarly we denote by N_n the set of all individuals within a group of size n, i.e. $N_n := \{1, 2, 3...n\}$. The vector of elements b_i for individuals belonging to group k of size $n^k \in \mathbb{N}$ is $\mathbf{b}^k \in (N_H)^{n_k}$. We define 1_n the vector of n elements, each taking value 1. Moreover we denote by $\mathbf{B}_n(b_i)$ the set of all $n-$dimensional vectors including the realization b_i for agent i and $\mathbf{B}_n(b_i, b_j) := \mathbf{B}_n(b_i) \cap \mathbf{B}_n(b_j)$, and more generally $\mathbf{B}_n(\mathbf{b}, b_t) := \mathbf{B}_n(\mathbf{b}) \cap \mathbf{B}_n(b_t)$. It is important to point out that even if functioning b are unidimensional (i.e. belong to the set N_H) they can embed multidimensional considerations, for instance each b can be considered as a list of achievements of an individual or a list of spaces where she is not-excluded. What we require in our analysis is a predefined order for the set of functionings.

Since the set of possible achievements within a group of size n (i.e. $\mathbf{B}_n$) is finite with cardinality H^n, any *transitive* and *complete* binary relation defined on $\mathbf{B}_n$ can be represent by an evaluation function. The result can be extended to hold for $\mathbf{B}$ since the set $\mathbf{B}$ is countable (see Kreps, 1988). We define $w(b_i; \mathbf{b})$ (dropping the group index k for expositional convenience) as the *freedom-of-choice based refined functioning.*

Definition 1 (FCRF: Freedom of Choice based Refined Functioning) *Let $b_i \in N_H$, $\mathbf{b} \in \mathbf{B}_n(b_i)$ for any $n \geq 1$. The function $w(b_i; \mathbf{b})$ such that $w : N_H \times (N_H)^n \rightarrow \mathbb{R}_+$ represents the functioning evaluation adjusted for the freedom of choice based on the opportunities faced by individuals belonging to a group whose distribution of functionings is given by vector* $\mathbf{b}$.

The function $w(b_i; \mathbf{b})$ characterizes individual well-being in terms of both *effectively realized outcomes* and *positive freedom of choice.*

We will derive axiomatically specifications of function $w(b_i; \mathbf{b})$ that are decomposable into two functions:

$v(b_i)$: *objective evaluation function*, related to the actual choice (it depends on individual's preferences, which are not explicit). We observe the concrete value of functioning indicator, and rank individuals in terms of that indicator (which can be evaluated by one or more variables).

$\phi(\mathbf{b})$: *evaluation function of freedom of choice*, related to the element of freedom linked with the objective discriminating characteristic (e.g. the sex, the class of age etc...). Thus we will obtain the specification

$$w(b_i; \mathbf{b}) = v(b_i) \cdot \phi(\mathbf{b}) \tag{1}$$

where $v(b_i)$ depends only on the functioning achieved by individual i and it is non-decreasing in b_i, that is $v(b) \geq v(b')$ if $b > b'$. Moreover function $\phi(\mathbf{b})$ is non-decreasing in each element of $\mathbf{b}$.

3.1 Axioms and characterizations

Now we introduce some axioms in order to describe some plausible properties that the function $w_k(.)$ should satisfy. Since we will focus on properties defined for members of a given group, in order to simplify the notation we will drop the subscript k from any FCRF.

The first two axioms introduce some basic properties regarding the relation between individuals within a group, in terms of achieved functioning. Let Π denote a permutation matrix and $\pi(i) : N_n \rightarrow N_n$ a permutation operator applied to index $i \in N_n$.

Axiom 1 ((A) Anonymity) *For all $b_i \in N_H$, all $\mathbf{b} \in \mathbf{B}_n(b_i)$ and all permutation matrices Π such that $b_i = b_{\pi(i)}$, then $w(b_i; \mathbf{b}) = w(b_{\pi(i)}; \Pi\mathbf{b})$.*

The anonymity property requires that function $w(b_i; \mathbf{b})$ ignores the names of individuals in the evaluation of achieved functioning, if individuals belong to the same subgroup.

Axiom 2 ((WMA) Weak Monotonicity in Achievements) *Let $b_i, b_j \in N_H$, if $b_i \geq b_j$ then $w(b_i; \mathbf{b}) \geq w(b_j; \mathbf{b})$ for all $\mathbf{b} \in \mathbf{B}_n(b_i, b_j)$.*

The axiom of weak monotonicity in terms of the achieved functioning requires that, given two individuals belonging to the same group, if the functioning b_i is higher than functioning b_j, then $w(b_i; \mathbf{b})$ will show a level of well-being at least as high as the well-being related to function $w(b_j; \mathbf{b})$.The comparison depends only on the achievement of individuals i and j, not on the achievements of the rest of agents belonging to the group.

Next property is crucial for the main characterizations of the FCRFs. It will induce the multiplicative separability formulation of w as in (1). We consider relative comparisons of FCRF of two distinct individuals i, j belonging to the same subgroup.

Axiom 3 ((SRI) Strong Relative Independence) *Let $b_i, b_j \in N_H$, and $\mathbf{b} \in \mathbf{B}_n(b_i, b_j)$:*

(i) If $w(b_i; \mathbf{b}) = w(b_j; \mathbf{b}) = 0$ then $w(b_i; \mathbf{b}') = w(b_j; \mathbf{b}') = 0$ for all $\mathbf{b}' \in \mathbf{B}_n(b_i, b_j)$.

(ii) If $0 < w(b_i; \mathbf{b}) \leq w(b_j; \mathbf{b})$ then

$$\frac{w(b_i; \mathbf{b})}{w(b_j; \mathbf{b})} = \frac{w(b_i; \mathbf{b}')}{w(b_j; \mathbf{b}')}$$

for all $\mathbf{b}' \in \mathbf{B}_n(b_i, b_j)$.

The first part of the axiom SRI tells us that there exists a minimum achievement level which is independent from the belonging to a certain group: if a person experiences an extremely negative situation (a serious disease, or the lack of all basic human rights), her condition cannot be compared or weighted with that of other individuals, even though they belong to the same group. The second part of axiom SRI claims that, for acceptably high levels of achievement, it is possible to compare the realization of individuals belonging to a given group: this comparison is *relative* and *independent from the group* the two individuals belong to.

In order to evaluate the effect of the freedom of choice component we suggest to consider an independence property requiring that the FCRF evaluation is consistent with expansion of groups achievements obtained adding new individuals to the group experiencing the same functioning realization.

Axiom 4 ((IEF) Independence from External Functionings) *Let* $b_0, b_i \in N_H$, $\mathbf{b}, \mathbf{b}' \in \mathbf{B}_n(b_i)$, $\mathbf{b}^1, \mathbf{b}^{1\prime} \in \mathbf{B}_{n+1}(b_i, b_0)$, *s.t.* $\mathbf{b}^1 = \mathbf{b} \cup b_0$ *and* $\mathbf{b}^{1\prime} = \mathbf{b}' \cup b_0$ *then* $w(b_i; \mathbf{b}) \geq w(b_i; \mathbf{b}')$ *if and only if* $w(b_i; \mathbf{b}^1) \geq w(b_i; \mathbf{b}^{1\prime})$.

Axiom IEF seems a natural property to suggest it requires that if an individual experience a larger FCRF once assigned to a group compared to another group then adding the same functioning level to the distribution of the two groups cannot induce a re-ranking in the comparison. As will turn out only some of the main FCRF we will characterize satisfy this property.

Next property serves to make comparison easier and to simplify the analysis. We consider the FCRF as a "membership function" whose realization quantifies the degree to which an individual achievement (also encompassing the freedom of choice component) contributes to experiencing fully a refined functioning. Thus we move from a level 0 of no experience of a given FCRF when all the group, including the individual experience the first level of the functioning, to level 1 when all the group experience the highest level of the functioning.

Axiom 5 ((N) Normalization) $w() = 1$ *and* $w() = 0$.

The normalization axiom N claims that if individual i achieves the highest level of refined functioning as do all the individuals in her reference group then $w(H; H \cdot \mathbf{1}_n) = 1$, while if she completely fails in reaching functioning, as do also all the individuals in her reference group then $w(1; \mathbf{1}_n) = 0$.

Next property will allow to extend comparisons over distributions of different population size.

Axiom 6 ((RI) Replication Invariance) $w(b_i; \mathbf{b}) = w(b_i; \mathbf{b}^r)$, *with* $\mathbf{b}^r$ *a vector of* r *replications of* $\mathbf{b}$.

The axiom RI tells that, if we consider two different groups, with one obtained by simply replicating r times each element of the other group, the value of function $w(.)$ remains unchanged. Given the vector $\mathbf{b} = (b_1, b_2, ..., b_n)$, the vector of functioning obtained by the replication procedure will be $\mathbf{b}^r = (\underbrace{b_1, b_1, ...}_{r}, \underbrace{b_2, b_2, ...}_{r}, \underbrace{b_n, b_n, ...}_{r})$

Axiom 7 ((WM) Weak Monotonicity) *Let* $\mathbf{b}, \mathbf{b}' \in \mathbf{B}_n(b_i)$ *if* $\mathbf{b} \geq \mathbf{b}'$, *then* $w(b_i; \mathbf{b}) \geq w(b_i; \mathbf{b}')$.

The axiom WM introduces the idea that there exists a relation between individual opportunities and the belonging to a certain subgroup: some groups assure to individuals a more valuable set of alternatives than others. Thus, the fact of belonging to this or that group may affect the value of function $w(.)$. More precisely, if group $\mathbf{b}$ offers to individual i more freedom of choice than group $\mathbf{b}'$, then the value of refined functioning $w(b_i; \mathbf{b})$ will be higher than $w(b_i; \mathbf{b}')$.

Next three properties introduce different criteria to evaluate the opportunities faced by an individual belonging to a certain group.

First property requires to consider improvement in opportunities only when the set of functioning experienced by the individuals belonging to an homogeneous group are enriched by a value above the previously existing maximum level. Let $\mathbf{b}^{\max}$ denote the highest functioning within the vector $\mathbf{b}$, i.e. $\mathbf{b}^{\max} := \max_{i \in N_n}\{b_i : b_i \in \mathbf{b}\}$

Axiom 8 ((MR) Maximal Rule) *Let* $b^*, b_i \in N_H$, $\mathbf{b} \in \mathbf{B}_n(b_i)$, $\mathbf{b}' \in \mathbf{B}_{n+1}(b_i, b^*)$, *s.t.* $\mathbf{b}' = \mathbf{b} \cup b^*$

(i) if $b^* > \mathbf{b}^{\max}$ *then* $w(b_i; \mathbf{b}') \geq w(b_i; \mathbf{b})$,

(ii) if $b^* \leq \mathbf{b}^{\max}$ *then* $w(b_i; \mathbf{b}') = w(b_i; \mathbf{b})$.

Axiom MR considers the best element observed within the subgroup as a target to evaluate the freedom of choice associated with the condition of belonging to that group: the higher the value of functioning observed in a subgroup, the higher the freedom of choice of individuals belonging to that group. The idea is that given a functioning level $\tilde{b} \in N_H$ the fact of belonging to a group where no agents reach at least the functioning value $\tilde{b}$ does constitute an objective limitation to the individual freedom of choice, if there are other groups showing $\tilde{b}$. The agent belonging to the first group has not all the potential to achieve high level of living standard.

Next Property TR requires to consider improvement in opportunities when the set of functioning experienced by the individuals belonging to an homogeneous group are enriched by a value strictly above a given threshold.

Axiom 9 ((TR)Threshold Rule) *Let* $b^*, b_i, z \in N_H$, $\mathbf{b} \in \mathbf{B}_n(b_i)$, $\mathbf{b}' \in \mathbf{B}_{n+1}(b_i, b^*)$, *s.t.* $\mathbf{b}' = \mathbf{b} \cup b^*$,

(i) if $b^* > z$ *then* $w(b_i; \mathbf{b}') \geq w(b_i; \mathbf{b})$,

(ii) if $b^* \leq z$ *then* $w(b_i; \mathbf{b}') = w(b_i; \mathbf{b})$.

Axiom TR requires that, in order to evaluate the opportunities faced by an individual, it is relevant to set a threshold z, and then assess the levels of achievements experienced within a group depending on whether, and by how many individuals the identified "critical level" is experienced. The threshold z can be seen as a target value of functioning, considered adequate by reasonable agents as an index of high living standard[4].

[4] For example: take the functioning "level of literacy". If the maximal value observed is the PhD level, yet we can choose the university degree level as a target, and discriminate individuals according to that threshold.

Finally, property TC requires that in order to evaluate the opportunities faced by an individual it is relevant to set an absolute threshold, and then assess the number of individuals belonging to the group exceeding that threshold compared with individuals below the threshold.

Axiom 10 ((TC) Threshold Consistence) *Let* $b_i, b_j, b'_i \in N_H$, $\mathbf{b} \in \mathbf{B}_n(b_i, b_j)$, $\mathbf{b}' \in \mathbf{B}_n(b'_i, b_j)$, $\mathbf{b}^*_{-i} \in \mathbf{B}_{n-1}(b_j)$ *s.t.* $\mathbf{b} = \mathbf{b}^*_{-i} \cup b_i$, $\mathbf{b}' = \mathbf{b}^*_{-i} \cup b'_i$,

(i) if either $[b_i \leq z$, *and* $b'_i \leq z]$, *or* $[b_i > z$, *and* $b'_i > z]$*then* $w(b_j; \mathbf{b}) = w(b_j; \mathbf{b}')$;

(ii) if $[b_i \leq z$, *and* $b'_i > z]$ *then* $w(b_j; \mathbf{b}') \geq w(b_j; \mathbf{b})$.

In other words the higher the number of agents exceeding the threshold in a subgroup compared with individuals below the threshold, the higher the freedom of choice. Thus if one individual below the threshold improves her situation and move above z, then the value of her FCRF cannot decrease. The TC axiom is stronger postulating that keeping the individual j performance fixed her FCRF cannot decrease also if someone else functioning achievement crosses the threshold. The idea is that the higher the number of individuals experiencing functioning levels exceeding threshold z the lower should be the effort we infer is required to individuals in the group qualified by the same characteristics in order to reach performance z.[5]

Our first result will provide the general class of FCRF based on some core axioms.

Lemma 1 *A FCRF satisfies A, WMA, SRI and RI if and only if*

(i) either there exist function $v : N_H \to \mathbb{R}_+$, *and* $\phi : n^{N_H} \to \mathbb{R}_{++}$ *such that*

$$w(b_i; \mathbf{b}) = v(b_i) \cdot \phi(\mathbf{b}) \tag{2}$$

for all $b_i \in N_H, \mathbf{b} \in \mathbf{B}_n(b_i)$, $n \geq 1$, *where* $v(.)$ *is non-decreasing,* $\phi(.)$ *is symmetric, replication invariant, and non-decreasing in each argument.*

(ii) or $w(b_i; \mathbf{b}) = 0$ *for all* $\mathbf{b} \in \mathbf{B}(b_i)$.

We now move to derive refinements of the result in the previous that further specify the functional for of ϕ. All next results hold for $H \geq 3$, if the analysis is restricted to binary functioning realizations then most of the property adopted will turn out to be redundant, see for instance Remark 2.

Proposition 1 *A FCRF satisfies A, WMA, SRI, RI and MR, N if and only if*

(i) either there exist function $v : N_H \to \mathbb{R}_+$, *and* $\psi : N_H \to \mathbb{R}_{++}$ *such that*

$$w(b_i; \mathbf{b}) = v(b_i) \cdot \psi(\mathbf{b}^{\max}) \tag{3}$$

[5] We can think of numbers (or proportions) of men and women occupying seats in the National Parliaments: in recent past (and still in most countries) the possibility to obtain seat in the Parliament was much more difficult for women than for men. This discrimination produced a numerical gap between sexes that can be used as an informative signal for our purposes.

for all $b_i \in N_H, \mathbf{b} \in \mathbf{B}_n(b_i)$, $n \geq 1$, *where* v *is non-decreasing,* ψ *is non-decreasing, where* $v(H) = 1/\psi(H)$, *and* $v(1) = 0$.

(ii) or $w(b_i; \mathbf{b}) = 0$ *for all* $\mathbf{b} \in \mathbf{B}(b_i)$.

Function $w(b_i; \mathbf{b})$ allows to refine the value of observed achieved functioning by a function $\psi(\mathbf{b}^{\max})$, which takes note of the freedom of choice within a group of homogeneous individuals. Proposition 1 suggests that the value of freedom of choice positively depends on the maximal level of functioning observed in a homogeneous group.

Considering the result of Lemma 1, axiom MR requires that $\phi(\mathbf{b}') \geq \phi(\mathbf{b})$ if $\mathbf{b}' = \mathbf{b} \cup b^*$ and $b^* > \mathbf{b}^{\max}$, otherwise $\phi(\mathbf{b}') = \phi(\mathbf{b})$ if $b^* \leq \mathbf{b}^{\max}$. Clearly the impact of axiom MR is mainly to reduce the admissible information in order to operationalize the idea of "wider" set of opportunities focussing attention only on maximal attainments in terms of functioning within the set of homogeneous individuals.

Remark 1 *The FCRF in (3) satisfies IEF.*

Remark 2 *If* $H = 2$ *the FCRF in (3) requires that:* $w(1;1) = 0$, $w(1;2) = 0$, $w(2;2) = 1$. *The result can be obtained making use of A, SRI and N.*

In order to state next proposition we need to introduce a transformation of the vector of achievements, let $\mathbf{b} \in \mathbf{B}$, and $z \in N_H$ then $\boldsymbol{\alpha}(\mathbf{b}, z)$ denotes the vector of elements $\alpha_i(\mathbf{b}, z)$ obtained such that

$$\alpha_i(\mathbf{b}, z) = \begin{cases} b_i & \text{if } b_i > z \\ z & \text{if } b_i \leq z \end{cases} . \tag{4}$$

That is all functionings in $\mathbf{b}$ that are not-higher than z are replaced by the value z.

While $\boldsymbol{\alpha}^T(\mathbf{b}, z)$ denotes the vector obtained from $\boldsymbol{\alpha}(\mathbf{b}, z)$ truncating all elements of value z. Note that if all elements $\boldsymbol{\alpha}(\mathbf{b}, z)$ are of value z (i.e. all functionings in $\mathbf{b}$ are not-higher than z) then $\boldsymbol{\alpha}^T(\mathbf{b}, z)$ does not exist, to take into account this event we make use of the following transformation of $\boldsymbol{\alpha}^T(\mathbf{b}, z)$:

$$\boldsymbol{\alpha}^{T*}(\mathbf{b}, z) = \begin{cases} \boldsymbol{\alpha}^T(\mathbf{b}, z) & \text{if } \boldsymbol{\alpha}^T(\mathbf{b}, z) \text{ exists} \\ z & \text{otherwise} \end{cases} . \tag{5}$$

Proposition 2 *Let* $z \in N_H$. *A FCRF satisfies A, WMA, SRI, RI, WM and TR, N if and only if*

(i) either there exist function $v : N_H \to \mathbb{R}_+$, *and* $\zeta : \mathbf{B} \to \mathbb{R}_{++}$ *such that*

$$w(b_i; \mathbf{b}) = v(b_i) \cdot \zeta(\boldsymbol{\alpha}^{T*}(\mathbf{b}, z)) \tag{6}$$

for all $\beta \in N_H, \mathbf{b} \in \mathbf{B}_n(\beta)$, $n \geq 1$, *with* $\boldsymbol{\alpha}^{T*}(\mathbf{b}, z)$ *derived in (5), where* v *is non-decreasing,* $v(H) = 1/\zeta(H)$,*and* $v(1) = 0$; *and* ζ *is symmetric, replication invariant, and non-decreasing.*

(ii) or $w(b_i; \mathbf{b}) = 0$ *for all* $\mathbf{b} \in \mathbf{B}(b_i)$.

Proposition 2 suggests that the value of freedom of choice positively depends on the existence of individuals exceeding a chosen threshold within a homogeneous group. It is possible to refine the value of observed achievement by combining the evaluation function $v(b_i)$ and a function $\zeta(\boldsymbol{\alpha}^{T*}(\mathbf{b}, z))$ which incorporates an information on freedom of choice.

Remark 3 *There exist FCRFs specified in (6) that do not satisfy IEF.*

Next proposition characterizes families of FCRFs where the group opportunity component is based on the evaluation of the proportion of individuals whose functioning is above a given threshold. That is denoting by $F_{\mathbf{b}}(\beta)$ the value of the distribution function of vector $\mathbf{b}$ evaluated at functioning β, the obtained evaluation will be expressed in term of the associated survival function $\bar{F}_{\mathbf{b}}(\beta) := F_{\mathbf{b}}(\beta)$ evaluated at some threshold level z, i.e. $\bar{F}_{\mathbf{b}}(z)$. Note that $\bar{F}_{\mathbf{b}} \in \mathbb{Z}_{[0,1]}$ where $\mathbb{Z}_{[0,1]}$ denotes the set of rational number between 0 and 1.

Proposition 3 *Let $z \in N_H$. A FCRF satisfies A, WMA, SRI, RI, N and TC if and only if*

(i) either there exist function $v : N_H \rightarrow \mathbb{R}_+$, and $\chi : N_H \times \mathbb{Z}_{[0,1]} \rightarrow \mathbb{R}_{++}$ such that

$$w(b_i; \mathbf{b}) = v(b_i) \cdot \chi(z, \bar{F}_{\mathbf{b}}(z)) \tag{7}$$

for all $b_i \in N_H, \mathbf{b} \in \mathbf{B}_n(b_i)$, $n \geq 1$, where v is non-decreasing, χ is non-decreasing, $v(1) = 0$ and $v(H) = 1/\chi(z, 1)$ if $z < H$ or $v(H) = 1/\chi(H, 0)$ if $z = H$.

(ii) or $w(b_i; \mathbf{b}) = 0$ for all $\mathbf{b} \in \mathbf{B}(b_i)$.

Proposition 3 claims that the value of freedom of choice positively depends on the proportion of individuals exceeding a functioning target within a homogeneous group. Function $w(.)$ refines the value of observed achievement by combining the evaluation function $v(b_i)$ and a function $\chi(z, \bar{F}_{\mathbf{b}}(z))$ which incorporates the information on freedom of choice.

Remark 4 *There exist FCRFs specified in (7) that do not satisfy IEF.*

We now move to consider the combined effect of the axioms applied in previous propositions. We will omit straightforward proofs.

Next result combines the effect of MR with TR. As a result the group evaluation should depend on $\boldsymbol{\alpha}(\mathbf{b}, z)^{\max}$, that is any increase in the maximum level of groups functionings below a threshold $z \in N_H$ does not improve groups opportunities. only once the maximum level of functionings experienced in a group is above the threshold it counts in identifying a potential improvement in opportunities.

Corollary 1 *Let $z \in N_H$. A FCRF satisfies A, WMA, SRI, RI, N, MR and TR if and only if*

(i) either there exist function $v : N_H \to \mathbb{R}_+$, and $\psi : N_H \to \mathbb{R}_{++}$ such that

$$w(b_i; \mathbf{b}) = v(b_i) \cdot \psi(\boldsymbol{\alpha}(\mathbf{b}, z)^{\max}) \tag{8}$$

for all $b_i \in N_H, \mathbf{b} \in \mathbf{B}_n(b_i)$, $n \geq 1$, where v is non-decreasing, ψ is increasing, where $v(H) = 1/\psi(H)$, and $v(1) = 0$.

(ii) or $w(b_i; \mathbf{b}) = 0$ for all $\mathbf{b} \in \mathbf{B}(b_i)$.

Next result identifies the combined effect of TC and TR. In order to derive the result we need to define an indicator function $I(\mathbf{b}, z)$ that reaches value 1 when at least an element of $\mathbf{b}$ is above z, while is 0 when all the elements of $\mathbf{b}$ are not above z :

$$I(\mathbf{b}, z) = \begin{cases} 1 & \text{if } \bar{F}_{\mathbf{b}}(z) > 0 \\ 0 & \text{if } \bar{F}_{\mathbf{b}}(z) = 0 \end{cases} . \tag{9}$$

The "freedom of choice" component for the rule we characterize will be sensitive only to the existence within the group of individual functioning levels above the threshold z. The value of the functionings above z and the proportion of individuals experiencing them will not play any role. In intuitive terms, the individual functioning valuation v will be scaled by a coefficient that respond positively to the fact that within the group at least some individuals have been able to achieve a functioning realization above the threshold. The value of the coefficient may also depend on the value of the threshold considered.

Corollary 2 *Let $z \in N_H$. A FCRF satisfies A, WMA, SRI, RI, N, TR,and TC if and only if*

(i) either there exist function $v : N_H \to \mathbb{R}_+$, and $\xi : N_H \times \{0, 1\} \to \mathbb{R}_{++}$ such that

$$w(b_i; \mathbf{b}) = v(b_i) \cdot \xi(z, I(\mathbf{b}, z)) \tag{10}$$

for all $b_i \in N_H, \mathbf{b} \in \mathbf{B}_n(b_i)$, $n \geq 1$, where v is non-decreasing, ξ is non-decreasing, $v(1) = 0$ and $v(H) = 1/\xi(z, 1)$ if $z < H$ or $v(H) = 1/\xi(H, 0)$ if $z = H$.

(ii) or $w(b_i; \mathbf{b}) = 0$ for all $\mathbf{b} \in \mathbf{B}(b_i)$.

The combined effect of all properties can be summarized in the following straightforward remark.

Remark 5 *The FCRFs in (10) satisfies MR and IEF.*

As a result the specification in Corollary 2 *satisfies all the axioms* listed above.

Inspection of the proof of Remark 4 can lead to the conclusion that

Remark 6 *The FCRFs specified in (7) satisfy IEF if and only if they are restricted to take the specifications in Corollary 2.*

While if we focus on the FCRFs specified in (6) then it is possible to identify a variety of functional forms for $\zeta(\boldsymbol{\alpha}^{T*}(\mathbf{b}, z))$ that allow to satisfy IEF. For instance this is the case for all increasing functions of the maximum or of the minimum value of $\boldsymbol{\alpha}^{T*}(\mathbf{b}, z)$. Note that this may not the case [as shown in the proof of Remark 3] if we consider increasing functions of some average of max and min of $\boldsymbol{\alpha}^{T*}$.

These last remarks show a theoretical point in favor of the results in Proposition 1 and Corollary 2 as plausible and attractive specifications for the FCRFs. The independence condition (IEF) we have assumed is really weak, since it requires to check consistency in across groups evaluations made concerning an individual with *the same functioning performance*. In essence the "freedom of choice" component effect cannot lead to re-rank of two groups prospects if the same functioning level is added to each group. The maximal rule focussing on the maximal functioning realization in a group seems a natural candidate for a specification of the FCRFs in Lemma 1 satisfying IEF. If the notion of threshold is also introduced then the indicator rule in Corollary 2 focussing on the existence of realizations above the threshold is a solution that also show the advantage of satisfying all the other axioms considered. Some intermediate rule in between those in Proposition 1 and Corollary 2 and still satisfying IEF can be conceived. For instance relying on the result in Proposition 2 it seems reasonable to suggest a *censored maximal rule* where differences in maximal values are relevant to assess the "freedom of choice" component only if these values are above a threshold or alternatively if they are all below the threshold.

3.1.1 Aggregate evaluations

A useful feature of all FCRFs rule in Lemma 1 is that they explicitly separate (in a multiplicative manner) the individual functioning valuation and the group valuation representing the freedom of choice component. As a result of this feature all aggregate population evaluations obtained averaging the individuals FCRFs are also separable in each group average of the individual functioning valuation.

For instance, considering K subgroups indexed by $k = 1, 2, .., K$, denoting by b_i^k the functioning of individual i belonging to group k, and by $\mathbf{b}^k$ the group distribution of the n^k individuals in group k, letting n the size of the whole population, if we apply the FCRFs rule $w(b_i; \mathbf{b}) = v(b_i^k) \cdot \phi(\mathbf{b}^k)$ derived in Lemma 1 we can formalize the aggregate functioning valuation adjusted in order to take into account the "freedom of choice" component as $W_\phi(\mathbf{b}) = \frac{1}{n}\sum_{i=1}^n w(b_i; \mathbf{b}) = \frac{1}{n}\sum_{i=1}^n v(b_i^k) \cdot \phi(\mathbf{b}^k)$. More precisely we have

$$\begin{aligned} W_\phi(\mathbf{b}) &= \sum_{k=1}^{K} \left(\frac{n^k}{n}\right) \cdot \phi(\mathbf{b}^k) \cdot \frac{1}{n^k} \sum_{i=1}^{n^k} v(b_i^k) \qquad (11) \\ &= \sum_{k=1}^{K} \left(\frac{n^k}{n}\right) \cdot \phi(\mathbf{b}^k) \cdot V(\mathbf{b}^k) \end{aligned}$$

where $V(\mathbf{b}^k) = \frac{1}{n^k}\sum_{i=1}^{n^k} v(b_i^k)$ denotes the average functioning valuation in group k.

The final formula in (11) is a weighted average of the $V(\mathbf{b}^k)$ components where each group average individual functioning valuation is weighted according to the group "freedom of choice" coefficient given by $\phi(\mathbf{b}^k)$ times the population share of the group n^k/n measuring the relevance of the group over the entire population.

Adapting the formula in (11) to the specifications of the FCRF in Proposition 1, Corollary 2 and Proposition 3 is simply a matter of adopting the relevant specification for the function ϕ. For each one of the mentioned specification we obtain, respectively:

$$\begin{aligned} W_\psi(\mathbf{b}) &= \sum_{k=1}^K \left(\frac{n^k}{n}\right) \cdot \psi(\mathbf{b}^{k\,\max}) \cdot V(\mathbf{b}^k) \\ W_\xi(\mathbf{b}) &= \sum_{k=1}^K \left(\frac{n^k}{n}\right) \cdot \xi(z, I(\mathbf{b}^k, z)) \cdot V(\mathbf{b}^k) \\ W_\chi(\mathbf{b}) &= \sum_{k=1}^K \left(\frac{n^k}{n}\right) \cdot \chi(z, \bar{F}_{\mathbf{b}^k}(z)) \cdot V(\mathbf{b}^k). \end{aligned}$$

4 Comparing different criteria

We compare the three different criteria proposed in Section 3.1 to evaluate refined functionings. Consider figure 1 where the cumulative distributions $\{F_A, F_B\}$ of individual functioning achievements for groups A and B are represented. Note that distribution A first order stochastically dominates distribution B since $F_A \leq F_B$ for all values of b. As is well known, for distributions of the same population size, this dominance is equivalent to the fact that distribution A can be obtained from B increasing the functioning realization of some individuals without affecting the achievement of the others. Clearly in terms of "freedom of choice" distribution A cannot lead to a lower evaluation than distribution B.

The criterion related to *maximal rule* (see Proposition 1) is sensitive to the evaluation of the *horizontal gap* between the maximal value observed in group A ($b_A^{\max}$), and the maximal value observed in group B ($b_B^{\max}$). The main advantage of this first criterion is that it is easy to understand and persuasive: a group of individuals showing a lower maximal functioning level compared to other groups, produces an objective limitation in terms of freedom of choice for individuals within the group, compared to the other group. The disadvantage of the criterion is that it concentrates solely on the upper point of the distribution without taking into account the rest of distribution as for instance is done by the first order stochastic dominance condition. In fact it may happen that most of individuals within a group show adequate levels of functioning, but none reaches the ultimately satisfactory level, while agents belonging to another group are located at the bottom of the achievement levels with few individuals showing extremely high performances. Also, in many empirical applications it happens that at least one individual in all compared groups reaches the same maximal level of functioning: in this case the maximal rule *becomes ineffective*. Moreover it is not able to discriminate between distributions where only few “lucky” individuals reach the top, compared to others where the maximal functioning achievement is a common feature of the population.

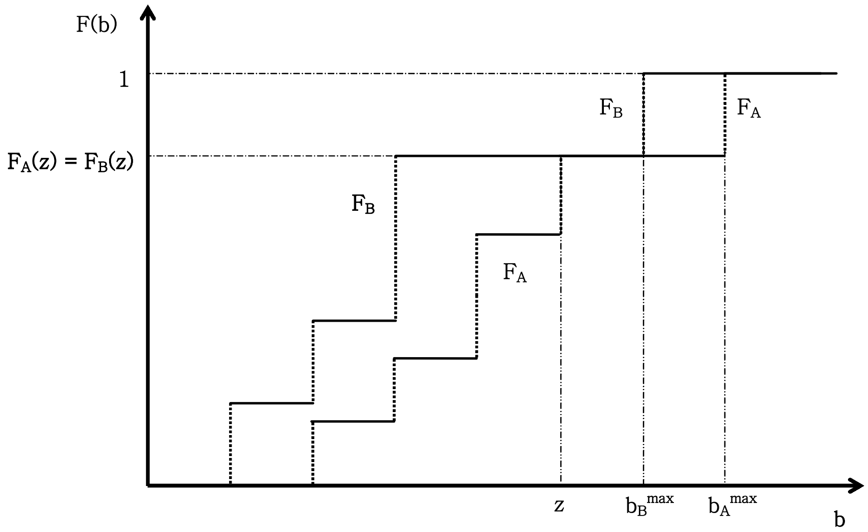

Fig. 1: Graphical representation of different criteria to evaluate FCRF

The criterion associated with the *threshold rule* (see Corollary 2) is similar to that related with the maximal rule: the main difference is that we consider a threshold not necessarily identical to the maximal observed value. Moreover we focus only on the information concerning the achievement of the threshold. In common with the maximal rule is the criticism that the threshold rule does not take into account neither how many people are above the threshold nor their distributions of the achieved functioning.

The criterion associated with the *threshold consistence rule* (see Proposition 2) on the other hand considers explicitly the proportion of population who exceeds a given threshold in terms of functioning. In figure 1, the distance in terms of “freedom of choice” between the two groups relates to the *vertical gap* between distribution of group A ($F_A(z)$) and distribution of group B ($F_B(z)$) given the chosen threshold. The main advantage of this criterion is that it embodies a different useful information about the distribution of different groups in terms of the functioning achievement. The disadvantage is that it is less obvious to understand in terms of capability: it is hard to think that the performance of two persons belonging to different groups must be valued differently, even when they achieve the same level of functioning and the groups show the same maximal functioning level.

To conclude all the criteria presented make use of useful and meaningful information, by their nature they focus on some specific features of the distribution of functionings. As the *maximal rule* turns out to be problematic because of it focuses only on maximal values forgetting the information on both all the other part of the distribution and the proportion of individuals at the maximum, the *threshold consistence rule* is concerned only on the proportion of population above a specified threshold, still disregarding the information on all the other part of the distribution. For expositional purpose in the figure we have depicted a situation where in terms of the threshold consistence rule the group performance is the same while the maximal rule favors distribution A, the graphs can be readjusted in order to show the opposite. Most importantly they can provide conflicting answers when applied to compare two distributions that are not ranked in term of first order stochastic dominance.

4.1 Examples

Consider the following case: suppose the population is partitioned into four subgroups ($k = 1, 2, 3, 4$) according to characteristic "class of age". We focus on the functioning *"health condition"* whose individual realizations are ordered into qualitative classes. We will evaluate individuals health condition using the FCRF function $w(b_i; \mathbf{b}) = v(b_i) \cdot \phi(\mathbf{b})$ derived in Lemma 1 and consider the implications related to the various specifications of $\phi(\mathbf{b})$ obtained in the previous sections. The functioning is evaluated by an ordinal indicator, taking values running from 1 to 6, where 1 means "very bad health condition" and 6 stands for "excellent health condition". We have four individuals $i \in \{A, B, C, D\}$ each one belonging respectively to subgroup 1,2,3 and 4.

The choosing sets are defined as follow:

$$\mathbf{b}^1 = \{2, 3, 3, 3, 4\}, \mathbf{b}^2 = \{1, 2, 4\}, \mathbf{b}^3 = \{1, 3, 4, 4, 4, 5\}, \mathbf{b}^4 = \{1, 2, 4, 6\} \quad (12)$$

where $b_A = 3 \in \mathbf{b}^1$, $b_B = 4 \in \mathbf{b}^2$, $b_C = 5 \in \mathbf{b}^3$, and $b_D = 4 \in \mathbf{b}^4$.

We compare the different criteria derived in Proposition 1, Corollary 2 and Proposition 3 in order highlight interesting features of their behavior and explain how they operationalize the notion of FCRF.

(*1*) First, we consider the *maximal rule*: $w(b_i; \mathbf{b}) = v(b_i) \cdot \psi_k(\mathbf{b}^{\max})$ derived in Proposition 1,

For expositional purposes we suggest an affine specification for the function $v(b_i)$ representing the individual functioning valuation and normalize its value such that $v(1) = 0$ and $v(H) = v(6) = 1$. Similarly we suggest an affine specification also for $\psi(\mathbf{b}^{\max})$ and require that the normalization conditions specified in Proposition 1 hold. In practice we let $v(b_i) = \alpha + \beta \cdot b_i$ where $\beta \geq 0$, and such that $v(1) = \alpha + \beta = 0$ and $v(H) = v(6) = \alpha + \beta \cdot H = \alpha + \beta \cdot 6 = 1$. As a result we obtain the following specification of function v :

$$v(b_i) = \frac{b_i - 1}{H - 1}. \quad (13)$$

While for function ψ we have $\psi(\mathbf{b}^{\max}) = \gamma + \delta \cdot b^{\max}$ where $\delta \geq 0$, and $\psi = \gamma + \delta \cdot b^{\max} > 0$. Applying the normalization condition in Proposition 1 we get

$\psi(H) = \gamma + \delta \cdot H = 1$, thus $\delta = (1-\gamma)/H$, which also implies that $\psi > 0$ if $\gamma + (1-\gamma)/H > 0$ i.e. $\gamma > -1/(H-1)$. For large values of H the condition $\gamma > -1/(H-1)$ can be approximated by $\gamma \geq 0$. That is we have

$$\psi_\gamma(\mathbf{b}^{\max}) := \gamma + (1-\gamma) \cdot \frac{\mathbf{b}^{\max}}{H}, \quad \gamma \in [0,1]\,. \tag{14}$$

The final parametric specification obtained for w is

$$w_\gamma(b_i;\mathbf{b}) = \frac{b_i - 1}{H-1} \cdot [\gamma + (1-\gamma) \cdot \frac{\mathbf{b}^{\max}}{H}] = \gamma \cdot \frac{b_i - 1}{H-1} + (1-\gamma) \cdot \frac{b_i - 1}{H-1} \cdot \frac{\mathbf{b}^{\max}}{H} \tag{15}$$

where $0 \leq \gamma \leq 1$.

Thus for $\gamma = 1$ we obtain only the individual functioning valuation, while on the other extreme for $\gamma = 0$ we get the FCRF $w_0(b_i;\mathbf{b}) = \frac{b_i-1}{H-1} \cdot \frac{\mathbf{b}^{\max}}{H}$ where the individual valuation is weighted by the relative maximal performance of the group in terms of functioning level $\frac{\mathbf{b}^{\max}}{H}$.

We consider (12) and derive the pattern of ranking for the 4 individuals depending on the value of γ. For this purpose we first compute $w_0(b_i;\mathbf{b}) = \frac{b_i-1}{H-1} \cdot \frac{\mathbf{b}^{\max}}{H}$ i.e. the refined functioning of the individuals evaluated for $\gamma = 0$ (see tab. 1). The ranking of the individuals will be obtained by comparing the values of $w_0^i := w_0(b_i;\mathbf{b})$.

Tab 1: An application of the *Maximal Rule*, with $\psi(\mathbf{b}^{\max}) = \frac{\mathbf{b}^{\max}}{H}$

Individual	$v(b_i) = \frac{b_i-1}{H-1}$	$\psi(\mathbf{b}^{\max}) = \frac{\mathbf{b}^{\max}}{H}$	$w_0(b_i;\mathbf{b}) = \frac{b_i-1}{H-1} \cdot \frac{\mathbf{b}^{\max}}{H}$
A	2/5	4/6	$2 \cdot 4/30 = 8/30$
B	3/5	4/6	$3 \cdot 4/30 = 12/30$
C	4/5	5/6	$4 \cdot 5/30 = 20/30$
D	3/5	6/6	$3 \cdot 6/30 = 18/30$

$$w_0^A < w_0^B < w_0^D < w_0^C\,.$$

The general result valid for $w_\gamma(b_i;\mathbf{b})$ can be obtained checking that the ranking associated with $w_1(b_i;\mathbf{b}) = v(b_i)$ is $w_1^A < w_1^B = w_1^D < w_1^C$.

Thus a convex combination of $w_1(b_i;\mathbf{b})$ and $w_0(b_i;\mathbf{b})$ as in (15) will lead to

$$\begin{aligned} w_\gamma^A &< w_\gamma^B < w_\gamma^D < w_\gamma^C \quad \text{for all } \gamma \in [0,1), \text{ and} \\ w_1^A &< w_1^B = w_1^D < w_1^C\,. \end{aligned}$$

(*2*) Next, we consider the *Threshold Rule* derived in Corollary 2: $\hat{w}(z, b_i;\mathbf{b}) = v(b_i) \cdot \xi(z, I(\mathbf{b}, z))$ where $I(\mathbf{b}, z)$ is an indicator function reaching value 1 when at least an element of $\mathbf{b}$ is above the threshold z, while $I(\mathbf{b}, z)$ is 0 when all the elements of $\mathbf{b}$ are not above z.

We set $v(b_i)$ as in (13), and set $z = 4$ the chosen threshold, identical for all subgroups. Moreover, inspired by part (1) of the example we set a parametric specification ξ_ρ of $\xi(z, I(\mathbf{b}, z))$ as an affine function i.e.

$$\xi_\rho(z, I(\mathbf{b}, z)) = \begin{cases} 1 & \text{if } \bar{F}_{\mathbf{b}}(z) > 0 \\ \rho & \text{if } \bar{F}_{\mathbf{b}}(z) = 0 \end{cases} = \rho + (1-\rho) \cdot I(\mathbf{b}, z), \tag{16}$$

where $\rho \in [0, 1]$ and $\bar{F}_{\mathbf{b}}(z)$ denotes the survival function of distribution $\mathbf{b}$ evaluated at z.

Note that for $\rho = 1$ all groups component receive the same evaluation equal to 1, while for $\rho = 0$ we get the maximal discrimination between groups depending on the existence of individuals experiencing functioning level above the threshold z. Thus we obtain

$$\hat{w}_\rho(z, b_i; \mathbf{b}) = \frac{b_i - 1}{H - 1} \cdot [\rho + (1-\rho) \cdot I(\mathbf{b}, z)] = \rho \cdot \frac{b_i - 1}{H - 1} + (1-\rho) \cdot \frac{b_i - 1}{H - 1} \cdot I(\mathbf{b}, z). \tag{17}$$

We derive the pattern of ranking of the 4 individuals depending on the value of ρ first computing $\hat{w}_0(z, b_i; \mathbf{b}) = \frac{b_i - 1}{H-1} \cdot I(\mathbf{b}, z)$ (see Tab. 2).

Tab 2: An application of the *Threshold Rule,* with $\xi(z, I(\mathbf{b}, z)) = I(\mathbf{b}, z)$ with $z = 4$.

Individual	$v(b_i) = \frac{b_i-1}{H-1}$	$\xi_0(4, I(\mathbf{b}, 4)) = I(\mathbf{b}, 4)$	$\hat{w}_0(4, b_i; \mathbf{b}) = \frac{b_i-1}{H-1} \cdot I(\mathbf{b}, 4)$
A	$2/5$	0	$2/5 \cdot 0 = 0$
B	$3/5$	0	$3/5 \cdot 0 = 0$
C	$4/5$	1	$4/5 \cdot 1 = 4/5$
D	$3/5$	1	$3/5 \cdot 1 = 3/5$

$$\hat{w}_0^A = \hat{w}_0^B < \hat{w}_0^D < \hat{w}_0^C$$

When $z = 4$, the general result valid for $\hat{w}_\rho(b_i; \mathbf{b})$ can be obtained checking that the ranking associated with $\hat{w}_1(b_i; \mathbf{b}) = v(b_i)$ is $\hat{w}_1^A < \hat{w}_1^B = \hat{w}_1^D < \hat{w}_1^C$.

Thus a convex combination of $\hat{w}_1(b_i; \mathbf{b})$ and $\hat{w}_0(b_i; \mathbf{b})$ as in (17) will lead to

$$\begin{aligned} \hat{w}_\rho^A &< \hat{w}_\rho^B < \hat{w}_\rho^D < \hat{w}_\rho^C \quad \text{for all } \rho \in (0, 1), \\ \hat{w}_0^A &= \hat{w}_0^B < \hat{w}_0^D < \hat{w}_0^C, \text{ and} \\ \hat{w}_1^A &< \hat{w}_1^B = \hat{w}_1^D < \hat{w}_1^C. \end{aligned}$$

Note that for any value of $z < 4$ there exist in any group at least one individual above the threshold, as a result $I(\mathbf{b}^k, z) = 1$ for any $k = 1, 2, 3, 4$ and any $z = 1, 2, 3$, thereby implying that for any $z = 1, 2, 3$ we have $\hat{w}_\rho(z, b_i; \mathbf{b}) = \hat{w}_1(z, b_i; \mathbf{b}) = v(b_i)$ for all $\rho \in [0, 1]$. While setting $z = 5$ will allow only to individual D to benefit from the higher value of the group component while all the other individuals will experience a value 0 for the indicator function $I(\mathbf{b}^k, z)$.

(*3*) Finally, we consider the third criterion, the Threshold Consistent Rule derived in Proposition 3: $\bar{w}(z, b_i; \mathbf{b}) = v(b_i) \cdot \chi(z, \bar{F}_{\mathbf{b}}(z))$.

We set $v(b_i)$ as in (13) and define a parametric specification χ_λ of $\chi(z, \bar{F}_{\mathbf{b}}(z))$ as an affine function of $\bar{F}_{\mathbf{b}}(z)$ normalized according to the prescriptions specified in Proposition 3. As a result of this procedure we obtain

$$\chi_\lambda(z, \bar{F}_{\mathbf{b}}(z)) = \lambda + (1-\lambda) \cdot \bar{F}_{\mathbf{b}}(z) \tag{18}$$

where $\lambda \in [0,1]$. Note that for $\lambda = 1$ all group components receive the same evaluation equal to 1, while for $\lambda = 0$ the group component of the FCRF is given by the group survival function evaluated at z.

Thus we get

$$\bar{w}_\lambda(z, b_i; \mathbf{b}) = \frac{b_i - 1}{H-1} \cdot [\lambda + (1-\lambda) \cdot \bar{F}_{\mathbf{b}}(z)] = \lambda \cdot \frac{b_i - 1}{H-1} + (1-\lambda) \cdot \frac{b_i - 1}{H-1} \cdot \bar{F}_{\mathbf{b}}(z). \tag{19}$$

Analogously to what done in the previous examples we evaluate $\bar{w}_0(z, b_i; \mathbf{b}) = \frac{b_i - 1}{H-1} \cdot \bar{F}_{\mathbf{b}}(z)$. Letting $z = 2$ we obtain:

Tab 3: An application of the *Threshold Consistent Rule,* with $\chi_0(z, \bar{F}_{\mathbf{b}}(z)) = \bar{F}_{\mathbf{b}}(z)$ with $z = 2$.

Individual	$v(b_i)$ $= \frac{b_i-1}{H-1}$	$\chi_0(2, \bar{F}_{\mathbf{b}}(z))$ $= \bar{F}_{\mathbf{b}}(2)$	$\bar{w}_0(2, b_i; \mathbf{b})$ $= \frac{b_i-1}{H-1} \cdot \bar{F}_{\mathbf{b}}(2)$
A	$2/5$	$4/5$	$8/25 = 0.32$
B	$3/5$	$1/3$	$1/5 = 0.2$
C	$4/5$	$5/6$	$2/3 = 0.6\bar{6}$
D	$3/5$	$2/4$	$3/10 = 0.3$

$$\bar{w}_0^B < \bar{w}_0^D < \bar{w}_0^A < \bar{w}_0^C$$

The ranking of $\bar{w}_0^i$ shows an improvement of agent A, even if her individual functioning valuation is the lower ($v^A = 0.4$) the refined functioning takes it almost completely into account ($\bar{w}_0^A = 0.32$) since 80% of individuals in group A are above the threshold. The general result valid for $\bar{w}_\lambda(b_i; \mathbf{b})$ can be obtained evaluating the convex combinations of $w_1(b_i; \mathbf{b}) = v(b_i)$ and $w_0(b_i; \mathbf{b})$ as in (18). Noting that the ranking of $\bar{w}_1^i$ is $\bar{w}_1^A < \bar{w}_1^B = \bar{w}_1^D < \bar{w}_1^C$ it is clear that the final ranking of $\bar{w}_\lambda(b_i; \mathbf{b})$ will depend on the values of λ.

We obtain $\bar{w}_0^B < \bar{w}_0^D < \bar{w}_0^C$ and $\bar{w}_0^A < \bar{w}_0^C$ for all $\lambda \in [0,1)$, while $\bar{w}_0^B < \bar{w}_0^A$ for $\lambda \in [0, 3/8)$ and $\bar{w}_0^D < \bar{w}_0^A$ for $\lambda \in [0, 1/11)$.

If instead we set $z = 4$, as in part 2 of the example, we get

Tab 4: An application of the *Threshold Consistent Rule,* with $\chi_0(z, \bar{F}_{\mathbf{b}}(z)) = \bar{F}_{\mathbf{b}}(z)$ with $z = 4$.

Individual	$v(b_i) = \frac{b_i - 1}{H-1}$	$\chi_0(4, \bar{F}_{\mathbf{b}}(z)) = \bar{F}_{\mathbf{b}}(4)$	$\bar{w}_0(4, b_i; \mathbf{b}) = \frac{b_i-1}{H-1} \cdot \bar{F}_{\mathbf{b}}(4)$
A	2/5	0	0
B	3/5	0	0
C	4/5	1/6	$4/30 = 0.1\bar{3}$
D	3/5	1/4	$3/20 = 0.15$

$$\bar{w}_0^B = \bar{w}_0^A < \bar{w}_0^C < \bar{w}_0^D$$

It is clear that the ranking in terms of $\bar{w}_\lambda^i$ will show individual D at the top only for low values of λ while in all other cases, it will place individual D above B and below C.

To summarize: In the example we obtain different rankings, according to different criteria applied. The result is not surprising since by construction among all 6 pairwise comparisons of the distributions $\mathbf{b}^k$ it is possible to identify a first order stochastic dominance relation between $\mathbf{b}^4$ and $\mathbf{b}^2$ and $\mathbf{b}^3$ and $\mathbf{b}^2$, for all other comparisons the stochastic dominance test fails. Given that the associated individual functioning values are $b_B = 4 \in \mathbf{b}^2 \leq b_D = 4 \in \mathbf{b}^4 < b_C = 5 \in \mathbf{b}^3$ then no matter what is the criterion applied we will always obtain that in terms of FCRFs $w^B \leq w^C$ and $w^B \leq w^D$.

For all other pairwise comparisons if the threshold consistent rule then *(i)* the choice of the threshold plays a fundamental role given that when two distributions cannot be ranked in terms of stochastic dominance then the ranking of their survival functions $\bar{F}_{\mathbf{b}}(z)$ depends on the value of z.

(ii) While the first two criteria focus on a target value and discriminate in a net way in terms of freedom of choice, the last (threshold consistent) rule is related to the distribution of individuals in each group but limits attention only at those above z.

Moreover, *(iii)* the choice of the refinement function $\phi(b)$ plays obviously a role. In the final section we will introduce some further assumptions in order to restrict the set of possible candidates for the functional forms of the "freedom of choice component".

5 Empirical analysis

The straightforward empirical application of the methodology described above is based on data from the micro-survey "Multiscopo" provided by Istat for the year 2002. This survey has not been designed *ad hoc* to assess well-being in the capability space: however, the analysis of data persuaded us that, despite the strong limitation in terms of variables, it can represent an adequate source of information to develop a battery of indicators and apply the refinement criteria. The survey is composed of more than 50.000 individuals.

The empirical analysis aims at clarifying some features of the procedures suggested in the previous sections. It concentrates on the functioning "*adequate cultural level*" of individuals older than 14 years. The functioning can be evaluated by using two ordinal indicators: an index of literacy and an indicator on the use of personal computer. The index of literacy is an ordinal variable which ranks from 1 (illiteracy) to 6 (university degree or more). The distribution of individuals according to this indicator is described in the following table 5:

Tab. 5: Literacy levels: distribution of population and gender-based distribution, (source: Istat survey "Multiscopo", 2002).

Literacy level	I_{1i}	$freq.$	$freq.$ $Male$	$freq.$ $Female$
Illiteracy	1	$5,6\%$	$3,5\%$	$7,6\%$
Primary school	2	$22,4\%$	$19,7\%$	$25,0\%$
Secondary school	3	$32,4\%$	$35,6\%$	$29,4\%$
3-years high school	4	$5,3\%$	$5,7\%$	$5,0\%$
5-years high school	5	$26,4\%$	$27,3\%$	$25,7\%$
university degree	6	$7,8\%$	$8,3\%$	$7,3\%$

About half of individuals aged 14 or more show a level of literacy equal to secondary school or lower: only 7,8 per cent of Italian population reaches a degree level. Some differences exists between men and women: the distribution of men clearly dominates (in terms of first order stochastic dominance) that of women in terms of literacy (data show that this is especially true among adult and old individuals).

PC clearly represents a powerful means of communication and transmission of culture, especially because it allows to access the web. The index on the use of PC reported in the Istat survey is an ordinal variable which ranks from 1 (use everyday) to 6 (use never): we can readjust it ranking performances in opposite order increasing in the use of PC. Table 6 shows the distribution of individuals according to this indicator.

Tab. 6: Use of PC: distribution of population and gender-based distribution, (source: Istat survey "Multiscopo", 2002).

Use of PC	I_{2i}	$freq.$	$freq.$ $Male$	$freq.$ $Female$
Never	1	$63,2\%$	$56,9\%$	$69,0\%$
Sometimes during the year	2	$1,5\%$	$1,4\%$	$1,6\%$
Sometimes every month	3	$2,7\%$	$2,4\%$	$3,0\%$
Once a week	4	$1,0\%$	$1,0\%$	$1,0\%$
Sometimes during the week	5	$10,8\%$	$12,0\%$	$9,6\%$
Every day	6	$20,8\%$	$26,3\%$	$15,8\%$

In 2002 the large part of Italian citizens (14 years old or more) still don't use personal computer: only one fifth of population uses PC every day. Moreover, large differences exist at the top of distribution between men and women: only 15,8 per cent of women switches on the PC every day, compared with 26,3 per cent of men. This is partly associated with the gender-based differences in the structure of Italian job market and the fact that most people use the PC as a every-day working tool. Note that also for this variable the "male distribution" first order stochastically dominates the "female distribution".

In order to arrive to the objective evaluation function $v(b_i)$ of functioning *adequate cultural level* we proceed by two steps: first we evaluate the mean value of the two indices for each individual, arriving to the functioning level b_i. Then we choose a function, linear in the level of achieved functioning: we can choose a straightforward normalization $v(b_i) = (b_i - 1)/(H - 1)$ as in (12). Our procedure assumes that the two indicators have the same relevance in the evaluation of functioning *adequate cultural level.*

Tab. 7: Functioning *adequate cultural level.* Distribution of population and gender-based distribution, (source: Istat survey "Multiscopo", 2002).

Functioning level	$b_i = \frac{I_{1i}+I_{2i}}{2}$	$freq.$	$freq.$ $Male$	$freq.$ $Female$	$v(b_i) = \frac{b_i-1}{H-1}$
Very bad	1	5,5%	3,5%	7,5%	0
bad	1,5	21,8%	18,7%	24,7%	0,1
bad-inadequate	2	21,8%	22,4%	21,2%	0,2
Inadequate	2,5	3,8%	3,9%	3,7%	0,3
sufficient	3	10,5%	9,2%	11,8%	0,4
Discrete	3,5	3,2%	3,1%	3,4%	0,5
Discrete-Good	4	5,7%	3,2%	5,2%	0,6
Good	4,5	5,9%	7,6%	4,3%	0,7
Very good	5	5,6%	5,9%	5,4%	0,8
Excellent	5,5	12,0%	14,5%	9,7%	0,9
Maximal	6	4,2%	5,2%	3,2%	1

Now we try to apply the criteria presented in Section 3.1, in order to arrive to a refined functioning of *adequate cultural level*, which takes note of an element of freedom of choice. We choose the gender as a relevant characteristic of freedom of choice: the aim is to test whether the belonging to a given gender-based group creates a discrimination in terms of freedom of choice. Note that also for the obtained distribution of the "averaged" functioning the "male distribution" first order stochastically dominates the "female distribution", recall that the dominance checked for the distributions of the two functionings I_1 and I_2 is not in general a sufficient condition for dominance in terms of their average.

We choose two representative individuals, which we call A and B.

A is a male, he is 30 and shows a level of $v(b_i) = 0,5$; B is a female, she is 30 and shows the same level of $v(b_i) = 0,5$. In terms of functioning level, A and B show the same result: however, as we can see in table 5 and 6, there exists a clear difference in the distribution of men and women in terms of achieved functioning. Our aim is to incorporate this difference in terms of opportunity, directly associated with the sex of individual, to the situation of individuals A and B.

Tab. 8: Evaluation of the level of functioning *adequate cultural level*. Representative individuals A and B (source: Istat survey "Multiscopo", 2002).

Individual	I_{1i}	I_{2i}	$b_i = \frac{I_{1i}+I_{2i}}{2}$	$v(b_i) = \frac{b_i-1}{H-1}$
A	4	3	$b_A = 3,5$	$0,5$
B	3	4	$b_B = 3,5$	$0,5$

First of all we apply the maximal rule. Let $w(b_i; \mathbf{b}) = v(b_i) \cdot b^{\max}/H$ be the freedom-of-choice based refinement function, which represents the most naïve case (already presented in the previous example: we weight the value of individual functioning $v(b_i)$ by multiplying it by $\mathbf{b}^{\max}/H$, which represents the maximum b_i observed in the subgroup normalized by its maximum feasible level H.

Tab 9: Application of the maximal rule (MR) with $\psi(\mathbf{b}^{\max}) = b^{\max}/H$. Representative individuals A and B (source: Istat survey "Multiscopo", 2002).

Individual	$v(b_i)$	$b^{\max}$	$w(b_i; \mathbf{b}) = \frac{b_i-1}{H-1} \cdot \frac{b^{\max}}{H}$
A	$0,5$	$b^{\max}_{male} = 6$	$0,5 \cdot 6/6 = 0,5$
B	$0,5$	$b^{\max}_{female} = 6$	$0,5 \cdot 6/6 = 0,5$

As we can notice, in both gender-based groups there exist at least one individual who shows the functioning level "6": the maximal rule thus becomes ineffective in discriminating between the performances of individual A and B in terms of FCRFs.

If we apply the *threshold rule*, when the FCRF is defined as in (17) and choose the threshold $z = 5$, identical for all subgroup, we obtain:

Tab 10: Application of the threshold rule (TR). Representative individuals A and B (source: Istat survey "Multiscopo", 2002).

Individual	$v(b_i)$	$I(\mathbf{b},5)$	$\hat{w}_0(5,b_i;\mathbf{b}) = \frac{b_i-1}{H-1} \cdot I(\mathbf{b},4)$
A	$0,5$	1	$0,5 \cdot 1 = 0,5$
B	$0,5$	1	$0,5 \cdot 1 = 0,5$

The threshold rule seems ineffective too: there exists at least one individual exceeding the threshold in both groups, no matter what is the level of z. The main reason for this result is associated with the fact that we are dealing with a large sample of representative individuals living in a country, and also probably because the suggested partition of the population is not sufficiently informative given the procedure we adopt.

Finally we apply the *threshold consistent rule*, setting the FCRF as defined in (18) and choose the threshold $z = 5$, identical for all subgroup.

Tab 11: An application of the threshold consistence (TC). Representative individuals A and B (source: Istat survey "Multiscopo", 2002).

Individual	$v(b_i)$	$F_{\mathbf{b}}(z)$	$w(b_i;b) = v(b_i) \cdot F_{\mathbf{b}}(z)$
A	$0,5$	19.7%	$0,5 \cdot 0,197 = 0,099$
B	$0,5$	12.9%	$0,5 \cdot 0,151 = 0,065$

In this last case we have a re-ranking: individual B belongs to the group of women that is strictly dominated by the group of men in terms of the freedom of choice component. Thus, the refined functioning of B must be weighted in order to take into account the higher effort required to women reach the same level of living standard.

It might be of interest to replicate the previous comparisons analyzing the effects deriving from further partitions of the groups. If also age is perceived to have an impact on identifying individuals capability sets in the space of cultural achievements then it might be appropriate to consider a further partition of the classes of males and female based on age. We consider three classes: individuals aged 18 and less, those aged 66 and more and the intermediate age class19-65. The distribution of the functioning for males and females is presented in the next two tables. From them it is possible to realize that it is not possible to rank all pairwise comparisons of groups in terms of stochastic dominance. In general middle age classes stochastically dominate older classes of the same sex while the younger class cannot be ranked if compared to the other two. The reason for this pattern is probably due to the obvious constraint imposed on the younger class that by definition cannot reach the higher educational levels. It seems that age plays a role in defining the opportunity sets in terms of the crude variable we have used to evaluate cultural achievements. Also sex plays a role since fixing the same class of age the "male distribution" stochastically dominates the "female distribution" for each class.

Tab. 12: Functioning *adequate cultural level.* Distribution based on class of age, *male.* (source: Istat survey "Multiscopo", 2002).

Functioning level	$v(b_i) = \frac{b_i-1}{H-1}$	*freq.* ≤ 18	*freq.* $19-65$	*freq.* ≥ 66
Very bad	0	-	1,3%	14,5%
bad	0,1	2,0%	12,7%	51,7%
bad-inadequate	0,2	15,7%	24,4%	15,1%
Inadequate	0,3	1,1%	4,3%	2,7%
sufficient	0,4	4,5%	9,8%	8,1%
Discrete	0,5	5,1%	2,7%	4,3%
Discrete-Good	0,6	35,4%	5,4%	0,7%
Good	0,7	35,2%	7,2%	0,5%
Very good	0,8	0,9	7,4%	0,5%
Excellent	0,9	0,1	18,3%	1,4%
Maximal	1	-	6,5%	0,6%

Tab. 13: Functioning *adequate cultural level.* Distribution based on class of age, *female.* (source: Istat survey "Multiscopo", 2002).

Functioning level	$v(b_i) = \frac{b_i-1}{H-1}$	*freq.* ≤ 18	*freq.* $19-65$	*freq.* ≥ 66
Very bad	0	0,1%	2,2%	26,1%
bad	0,1	1,9%	17,6%	52,4%
bad-inadequate	0,2	20,9%	24,6%	10,2%
Inadequate	0,3	2,7%	4,3%	1,8%
sufficient	0,4	7,5%	13,4%	7,1%
Discrete	0,5	5,9%	3,7%	1,9%
Discrete-Good	0,6	38,8%	4,7%	0,2%
Good	0,7	21,9%	4,5%	0,1%
Very good	0,8	0,22%	7,3%	0,1%
Excellent	0,9	-	13,3%	0,1%
Maximal	1	-	4,4%	0,1%

Next tables list the group components to use to supplement the individual functioning valuations v in order to derive the various formulations for FCRFs. Table 14 provides the relevant information for computing, according to the parametric specifications derived in the previous examples, the maximal rule w_0, the threshold rule $\hat{w}_0$with thresholds sets at $z = 4.5$ and $z = 5$ and the threshold consistent rule $\bar{w}_0$ also with thresholds sets at $z = 4.5$ and $z = 5$.

Tab. 14: Evaluation of the level of functioning *adequate cultural level.* partition age/sex (source: Istat survey "Multiscopo", 2002). Some group components.

Individual	$\mathbf{b}^{\max}/H$	$I(5,\mathbf{b})$	$I(4.5,\mathbf{b})$	$F_{\mathbf{b}}(5)$	$F_{\mathbf{b}}(4.5)$
$male \leq 18$	$0,9$	1	1	$0,1\%$	$1,1\%$
$male19-65$	1	1	1	$24,8\%$	$32,2\%$
$male \geq 66$	1	1	1	$2,0\%$	$2,5\%$
$female \leq 18$	$0,8$	0	1	0	$0,22\%$
$female19-65$	1	1	1	$17,7\%$	$25,0\%$
$female \geq 66$	1	1	1	$0,2\%$	$0,3\%$

Next tables make uses of the results in Tab 14 and specifies some particular values for the group component (14), (16) and (18) respectively associated with the parametric FCRF in (15), (17) and (19) evaluated for the parameter levels set at 0.5.

Tab. 15: Evaluation of the level of functioning *adequate cultural level.* partition age/sex (source: Istat survey "Multiscopo", 2002). Intermediate evaluations of parametric group components.

Individual	$\psi_{0.5}$	$\xi_{0.5}[5]$	$\xi_{0.5}[4.5]$	$\chi_{0.5}[5]$	$\chi_{0.5}[4.5]$
$male \leq 18$	$0,95$	1	1	$0,500$	$0,506$
$male19-65$	1	1	1	$0,624$	$0,661$
$male \geq 66$	1	1	1	$0,510$	$0,513$
$female \leq 18$	$0,9$	0.5	1	$0,500$	$0,501$
$female19-65$	1	1	1	$0,589$	$0,625$
$female \geq 66$	1	1	1	$0,501$	$0,503$

6 Extending the analysis

As pointed out earlier in order to rank individuals in terms of FCRFs the result associated with the characterizations obtained in Section 3.1 are clearly sensitive to the choice of the function $\phi(\mathbf{b})$ evaluating the "freedom of choice component".

In order to derive robust (i.e. independent from the functional form of $\phi(\mathbf{b})$) explicit answers to comparisons in terms of FCRF of individuals belonging to different groups it can be shown that it is necessary that one individual dominates the other both in terms of individual functioning b and in terms of group functionings distributions $\mathbf{b}$. If $\phi(\mathbf{b})$ is specified according to the *maximal rule* or the *threshold rule* derived in Section 3.1 then it is required that the dominant individual belongs to a group that dominates the other in terms of $\mathbf{b}^{\max}$, while if we adopt the *threshold consistency rule* in order to reach a robust ranking it is still required stochastic dominance in terms of group functionings distribution $\mathbf{b}$.

In order to increase the power of the comparison test next property suggests a natural weak extension of the between group comparability hypothesis assuming that for an individual FCRF an increase of "one step" in his functioning level can compensate for a decrease of "one step" in the functioning achievement of another individual belonging to the same group.

Axiom 11 ((WIP) Weak Individual Priority) *Let* $b_j, b_i \in N_H, b_i > 1, b_j < H$, *and* $\mathbf{b} \in \mathbf{B}_n(b_i, b_j)$, $\mathbf{b}' \in \mathbf{B}_n(b_i - 1; b_j + 1)$, *s.t.* $\mathbf{b}' = \mathbf{b}_{-\{i,j\}} \cup (b_i - 1; b_j + 1)$ *then* $w(b_i; \mathbf{b}) \geq w(b_i - 1; \mathbf{b}')$.

The priority assumption underlying WIP may have different effects on the implementation of between groups comparisons depending on the criteria considered. We analyse these implications for the FCRF, associated with the *maximal rule*, derived in Proposition 1 i.e. for $w(b_i; \mathbf{b}) = v(b_i) \cdot \psi(\mathbf{b}^{\max})$ where v is non-decreasing, ψ is non-decreasing, and normalization leads to $v(H) = 1/\psi(H)$ and $v(1) = 0$. Related results can be obtained for the *threshold rule*, while the *threshold consistency rule* may require a stronger specification of the individual priority property.

Lemma 2 *The FCRFs in (3) satisfy WIP if and only if*

$$\min_{m \geq b > \underline{b}+1} \left\{ \frac{v(b)}{v(b-1)} \right\} \geq \frac{\psi(m+1)}{\psi(m)} \text{ for any } m \leq H - 1 \qquad (20)$$

where $\underline{b} := \max\{b \in N_H : v(b) = 0\}$.

The result identifies restrictions on the values of the function ψ related to the "group component" conditional on the behavior of v related to the "individual achievements". Note that since v is non-decreasing then $v(b)/v(b-1) \geq 1$ for all $b > \underline{b} + 1$, and given that also ψ is non-decreasing, from (20) it has to be $\psi(m+1)/\psi(m) \geq 1$ for all $m \in \{\underline{b} + 1, ..., H - 1\}$. Therefore if there exists b such that $v(b) = v(b-1)$ then $\psi(m) = \psi(b)$ for all $m \geq b$, i.e. the function ψ is constant for all values higher than b.

The result in Lemma 2 is still too general to allow to make the criterion operational without specifying appropriate functional forms for the functions v and ψ. However, in some cases, once a family of functions v is identified it is possible to derive more precise specifications also for the associated FCRFs making use of Lemma 2 to narrow down the set of admissible ψ. This is the case for the family $v_{\{\underline{b},\bar{b}\}}(b)$ of "membership functions" used in empirical and theoretical analysis [see Cerioli and Zani (1990), Brandolini and D'Alessio (2000) and Chiappero (2000)]. The parametric family $v_{\{\underline{b},\bar{b}\}}(b)$ is specified as

$$v_{\{\underline{b},\bar{b}\}}(b) = \begin{cases} 0 & \text{if } 1 \leq b \leq \underline{b} \\ \frac{b-\underline{b}}{\bar{b}-\underline{b}} & \text{if } \underline{b} < b \leq \bar{b} \\ 1 & \text{if } \bar{b} < b \leq H \end{cases} \qquad (21)$$

where $\underline{b} < \bar{b} \in N_H$, it associates equal increments to the value of v to each increment in the classes of individual achievements b censoring the values of the function above $\bar{b}$ and below $\underline{b}$. Note that the valuation function suggested in the examples is a special case of $v_{\{\underline{b},\bar{b}\}}$ obtained for $\underline{b} = 1$ and $\bar{b} = H$.

Our aim is to investigate a possible test for dominance between individuals based on (21) for fixed $1 \leq \underline{b} < \bar{b} \leq H$ and made robust to the choice of the admissible ψ satisfying WIP according to Lemma 2. We denote by $\Psi(\underline{b}, \bar{b})$ the set of all admissible ψ consistent with $v_{\{\underline{b},\bar{b}\}}$ as (21) according to the conditions specified in Lemma 2.

Next result turns out to provide a clear-cut answer to our question identifying a simple set of testable conditions for dominance expressed in terms of functionings b. For this purpose we define by $b_{[\underline{b},\bar{b}]}$ the value of b censored at $\bar{b}$ for values above $\bar{b}$ and censored at $\underline{b}$ for values below $\underline{b}$, similarly we define by $m_{[z]}$ a function where the value of is censored at z for values below z.

Proposition 4 *Let* $b, b' \in N_H$,*and* $\mathbf{b} \in \mathbf{B}(b)$, $\mathbf{b}' \in \mathbf{B}(b')$, *then*

$$v_{\{\underline{b},\bar{b}\}}(b) \cdot \ \psi(\mathbf{b}^{\max}) \geq v_{\{\underline{b},\bar{b}\}}(b') \cdot \ \psi(\mathbf{b}'^{\max})$$

for all $\psi \in \Psi(\underline{b}, \bar{b})$ *if and only if:*

(i) $b_{[\underline{b},\bar{b}]} \geq b'_{[\underline{b},\bar{b}]}$ *and*

(ii) $\max\{b - \underline{b}; 0\} \cdot \min\left\{[\mathbf{b}^{\max} - 1 - \underline{b}]_{[1]}; \bar{b} - \underline{b}\right\} \geq \max\{b' - \underline{b}; 0\} \cdot \min\{[\mathbf{b}'^{\max} - 1 - \underline{b}]_{[1]}; \bar{b} - \underline{b}\}$.

In particular if both $\underline{b} \leq b', b \leq \bar{b}$ and $2 \leq \mathbf{b}^{\max}, \mathbf{b}'^{\max} \leq \bar{b} + 1$ the two dominance conditions in the proposition can be specified as:

(i) $b \geq b'$ and

(ii) $(b - \underline{b}) \cdot (\mathbf{b}^{\max} - 1 - \underline{b}) \geq (b' - \underline{b}) \cdot (\mathbf{b}'^{\max} - 1 - \underline{b})$.

While condition (i) requires dominance in terms of individual functioning condition (ii) does not necessarily require dominance in terms of maximal group value, it is still possible that (ii) holds while $\mathbf{b}^{\max} < \mathbf{b}'^{\max}$.

The result suggests to adopt the following a possible specification of the maximal rule when $\underline{b} = 1$ and $\bar{b} = H > 2$ as required in the examples in the previous section:

$$w(b_i; \mathbf{b}) = \frac{b_i - 1}{H - 1} \cdot \frac{\max\{b^{\max} - 2; 1\}}{H - 2} \tag{22}$$

which is normalized such that $w(H; H \cdot \mathbf{1}) = 1$.

7 Conclusion

The aim of this work is mainly a tentative to define a criterion for a more comprehensive way to evaluate well-being, which takes into account an information regarding individual freedom of choice. We have described the capability approach, a powerful theoretical framework for the definition of well-being defended by the Nobel A.K. Sen and suggested a methodology to make operationalize the notion of capability set. Once the population is partitioned into subgroups of homogeneous individuals then if the partition is appropriately defined the variability in the space of functioning achievements can be interpreted as being related to individual choices and not to their relevant characteristics. As a result the group distribution will provide information on the individual capability set.

We try to justify the choice of the capability approach as a theoretical basis for a more comprehensive assessment of individuals' quality of life. We provides a formal characterization of a *freedom-of-choice based refinement function (FCRF)*, a function capable to evaluate both individuals' achievement and their freedom of choice. The specification of some rules that formalize thee *freedom-of-choice based refinement function* is axiomatically derived. Some examples helped to explain how the function works and a straightforward empirical application is presented based on a large survey provided by Istat for 2002. To conclude we suggest a possible procedure to make robust individual comparisons based on FCRFs

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A Appendix

Proof of Lemma 1. *Necessity part.* Consider axiom A, let $b_i = \beta$ and denote by $\hat{\mathbf{b}}$ the vector of n elements obtained ranking the elements of $\mathbf{b}$ in non-decreasing order. Applying A we get $w(b_i, \mathbf{b}) = w(\beta, \hat{\mathbf{b}})$, thus what is relevant are the elements of vector $\mathbf{b}$ and not their order.

By WMA we have that if $\beta' > \beta$ then $w(\beta', \hat{\mathbf{b}}) \geq w(\beta, \hat{\mathbf{b}})$ for all ranked distributions $\mathbf{b}$ whose elements include β' and β.

Given a function w then we let $Z(\mathbf{b}, w)$ denote the set of all values of $\mathbf{b}$ such that $w(.; \mathbf{b}) = 0$, that is $Z(\mathbf{b}, w) := \{b_i : w(b_i; \mathbf{b}) = 0\}$. Note that $Z(\mathbf{b}, w)$ might also be empty if w is always positive. Moreover, if there exist β' such that $w(\beta'; \hat{\mathbf{b}}) = 0$, then by WMA it is also true that $w(\beta; \hat{\mathbf{b}}) = w(\beta'; \hat{\mathbf{b}}) = 0$ for all $\beta < \beta'$ such that β belongs to vector $\mathbf{b}$.

Consider SRI, then according to part (i) if $w(\beta; \hat{\mathbf{b}}) = 0$ then $w(\beta; \mathbf{b}') = 0$ for all $\mathbf{b}' \in \mathbf{B}_n(\beta)$.

While part (ii) of SRI, in conjunction with A, requires that, if $w(\beta; \mathbf{b}) > 0$, then

$$\frac{w(\beta; \mathbf{b})}{w(\beta'; \mathbf{b})} = \frac{w(\beta; \mathbf{b}')}{w(\beta'; \mathbf{b}')} \iff \frac{w(\beta; \hat{\mathbf{b}})}{w(\beta; \hat{\mathbf{b}}')} = \frac{w(\beta'; \hat{\mathbf{b}})}{w(\beta'; \hat{\mathbf{b}}')}.$$

It follows that the ratio $w(\beta; \mathbf{b})/w(\beta'; \mathbf{b})$ is independent from the vector $\mathbf{b}$ provided that it includes the values β, β' that is, if $w(\beta; \mathbf{b}) > 0$ then

$$\frac{w(\beta; \mathbf{b})}{w(\beta'; \mathbf{b})} = f(\beta, \beta') \text{ for all } \mathbf{b} \in \mathbf{B}_n(\beta, \beta'), \text{ all } \beta' \geq \beta \in N_H. \tag{23}$$

We let $n = 2H$, and focus on a specific vector of achievements, denote by

$$\hat{\mathbf{b}}_0 := (1, 1, 2, 2, 3, 3, .., H-1, H-1, H, H)$$

the ordered vector where any individual exhibit a different level of achievement and each achievement appears twice. It follows that $\hat{\mathbf{b}}_0 \in \mathbf{B}_{2H}(\beta, \beta')$, therefore SRI and A require that

$$\frac{w(\beta; \mathbf{b}_0)}{w(\beta'; \mathbf{b}_0)} = f(\beta, \beta')$$

moreover, since $\mathbf{b}_0$ includes all possible values of achievements appearing twice then letting $\beta'' \neq \beta'$ we have also that

$$\frac{w(\beta; \mathbf{b}_0)}{w(\beta''; \mathbf{b}_0)} = f(\beta, \beta'').$$

In general, if $n = 2H$ the function $f(\beta, \beta')$ coincides with $w(\beta; \mathbf{b}_0)/w(\beta'; \mathbf{b}_0)$. As a result SRI and A require that for a given $\mathbf{b} \in \mathbf{B}_n(\beta, \beta')$

$$\frac{w(\beta; \mathbf{b})}{w(\beta'; \mathbf{b})} = \frac{w(\beta; \mathbf{b}_0)}{w(\beta'; \mathbf{b}_0)} \text{ for all } \beta' \geq \beta \in N_H$$

that is

$$w(\beta; \mathbf{b}) = w(\beta; \mathbf{b}_0) \cdot \frac{w(\beta'; \mathbf{b})}{w(\beta'; \mathbf{b}_0)} \text{ for all } \beta' \geq \beta \in N_H \tag{24}$$

since $\mathbf{b}_0$ is fixed then we can redefine $v(\beta) := w(\beta; \mathbf{b}_0)$, moreover since (24) has to hold for all $\beta' \geq \beta \in N_H$ then for a given value of β the second term on the right hand side of (24) has to be independent from β'. That is $w(\beta'; \mathbf{b})/v(\beta')$ should not depend from β', but it can still in principle depend from β which is the lower bound for the β' values, and of course can depend from $\mathbf{b}$. Thus we have

$$\frac{w(\beta'; \mathbf{b})}{w(\beta'; \mathbf{b}_0)} = g(\beta; \mathbf{b}) \text{ for all } \beta' \geq \beta \in N_H \tag{25}$$

however, unless β' is the minimum value of the achievements in $\mathbf{b}$ there can possible be plenty of potential values of β that are consistent with $\beta' \geq \beta$ thus, setting $\beta'' = \min(\mathbf{b})$ such $w(\beta''; \mathbf{b}) > 0$ we get that restating (24) substituting β'' for β we obtain that (25) has to hold for all $\beta' \geq \beta''$, for β'' possibly different from β thus $g(\beta; \mathbf{b})$ has to be independent from β giving

$$w(\beta; \mathbf{b}) = v(\beta) \cdot \phi(\mathbf{b}) \text{ for all } \beta \in N_H, \mathbf{b} \in \mathbf{B}_{2H}(\beta) \tag{26}$$

such that $w(\beta; \mathbf{b}) > 0$.

In order to extend the result to all distributions such that $n \geq 2$ we make use of the RI axiom. That is for any $\mathbf{b} \in \mathbf{B}_n(\beta)$ there exist a $\mathbf{b}' \in \mathbf{B}_{2H}(\beta)$ such that $(\mathbf{b})^{2H} = (\mathbf{b}')^n$ where $(\mathbf{b})^r$ denotes a r times replication of vector $\mathbf{b}$. According to RI, if $w(\beta; \mathbf{b})$ satisfies A, we have that

$$w(\beta; \mathbf{b}) = w(\beta; \hat{\mathbf{b}}) = w(\beta; (\hat{\mathbf{b}})^{2H}) = w(\beta; (\hat{\mathbf{b}}')^n) = w(\beta; \hat{\mathbf{b}}') = v(\beta) \cdot \phi(\mathbf{b}') \tag{27}$$

where $\mathbf{b}$ and $\mathbf{b}'$ are made equivalent applying permutations and replications of element. That is denoting with $F(\mathbf{b})$ the distribution function of vector $\mathbf{b}$, we have that $F(\mathbf{b}) = F(\mathbf{b}')$ in (27). Thus for all $n \geq 1$, letting $\tilde{\phi}(F(\mathbf{b})) := \phi(\mathbf{b})$ we have

$$w(\beta; \mathbf{b}) = v(\beta) \cdot \tilde{\phi}(F(\mathbf{b})) \tag{28}$$

if $w(\beta; \mathbf{b}) > 0$. Recall that by WMA $v(\beta)$ is non-decreasing in β.

To conclude we show that applying SRI part (i), A and RI we have that $w(\beta; \mathbf{b}) = 0$ where $\mathbf{b} \in \mathbf{B}_n(\beta)$ if and only if $w(\beta; \mathbf{b}_0) = 0$ where $\mathbf{b}_0 \in \mathbf{B}_{2H}(\beta)$. Making use or RI there exists $\mathbf{b}' \in \mathbf{B}_n(\beta)$ s.t. $\mathbf{b}'$ and $\mathbf{b}_0$ can be replicated respectively according to the factors $2H$ and n such that the obtained distributions belong to $\mathbf{B}_{2H \cdot n}(\beta)$. Making use of SRI part (i) $w(\beta; (\mathbf{b}_0)^n) = 0 \leftrightarrow w(\beta; (\mathbf{b}')^{2H}) = 0$, then by RI we have $w(\beta; (\mathbf{b}_0)^n) = 0 = w(\beta; \mathbf{b}_0)$ and $w(\beta; (\mathbf{b}')^{2H}) = 0 = w(\beta; \mathbf{b}')$, by applying SRI to $w(\beta; \mathbf{b}') = 0$, we get $w(\beta; \mathbf{b}') = 0 \leftrightarrow w(\beta; \mathbf{b}) = 0$, thus $w(\beta; \mathbf{b}_0) = 0 \leftrightarrow w(\beta; \mathbf{b}) = 0$.

Recalling that we have defined $v(\beta) := w(\beta; \mathbf{b}_0)$ then we can complete the result by extending (28) also to the cases where $w(\beta; \mathbf{b}) = 0$, obtained letting $v(\beta) = 0$.

Note that $\phi(\mathbf{b}) = 0$ for some $\mathbf{b} \in \mathbf{B}$ implies $w(\beta; \mathbf{b}') = 0$ for all $\mathbf{b}' \in \mathbf{B}(\beta)$. Apply A, suppose there exist β s.t. $v(\beta) > 0$ and $\phi(\beta \mathbf{1}_1) = 0$, then by RI $\phi(\beta \mathbf{1}_n) = 0$ for any $n \geq 1$. Let $n \geq 3$, by SRI (part i) $\phi(\mathbf{b}) = 0$ for all $\mathbf{b} \in \mathbf{B}_n(\beta, \beta)$ obtained modifying the achievements of all individuals except two. Consider $\mathbf{b}'' = (\beta, \beta, \beta'') \in \mathbf{B}_3(\beta, \beta)$, note that $w(\beta''; \mathbf{b}'') = 0 = w(\beta; \mathbf{b}'')$ since $\mathbf{b}'' \in \mathbf{B}_3(\beta, \beta)$ and thus $\phi(\mathbf{b}'') = 0$. Applying again SRI we get $\phi(\mathbf{b}) = 0$ for all $\mathbf{b} \in \mathbf{B}_3(\beta, \beta'')$. Consider now $\mathbf{b}''' = (\beta, \beta''', \beta'') \in \mathbf{B}_3(\beta, \beta'')$, repeating the same arguments followed $\phi(\mathbf{b}'') = 0$ we obtain that $\phi(\mathbf{b}''') = 0$ and applying again SRI we obtain $\phi(\mathbf{b}) = 0$ for all $\mathbf{b} \in \mathbf{B}_3(\beta''', \beta'')$. Note that vectors $\mathbf{b} \in \mathbf{B}_3(\beta''', \beta'')$ do not necessary include realizations of achievements β. Repeated application of the procedure allows to conclude that $\phi(\mathbf{b}) = 0$ for all $\mathbf{b} \in \mathbf{B}_3$. Similar result can be constructed for all $\mathbf{b} \in \mathbf{B}_n$ where $n \geq 3$. Applying RI the result $\phi(\mathbf{b}) = 0$ can hold for any $\mathbf{b} \in \mathbf{B}$.

Note now that $w(b_i; \mathbf{b}) = 0$ for all $\mathbf{b} \in \mathbf{B}(b_i)$ is a solution of the problem since satisfies all axioms.

To complete the proof we add WM. Axiom WM requires that $\phi(\mathbf{b})$ is non-decreasing in each component of $\mathbf{b}$.

The *sufficiency part* of the proof requires to check that (2) satisfies A, WM, WMA, RI, and SRI. ■

Proof of Proposition 1. Notice that applying MR (part ii) we obtain that $\phi(\mathbf{b}') = \phi(\mathbf{b})$ for all $\mathbf{b}, \mathbf{b}' \in \mathbf{B}(\beta, H)$ and for all $\beta \leq H$, since any additional elements cannot be larger than H. Setting $\beta = H$ we obtain that $\phi(\mathbf{b}') = \phi(\mathbf{b})$ for all $\mathbf{b}, \mathbf{b}' \in \mathbf{B}(H)$. The set $\mathbf{B}(H)$ identifies an equivalence class composed by all vectors where at least one element has value H. Consider the set of vectors $\mathbf{B}(H-1) \backslash [\mathbf{B}(H-1) \cap \mathbf{B}(H)]$, it is also an equivalence class because adding elements with value not larger than $H-1$ to any vector in $\mathbf{B}(H-1) \backslash [\mathbf{B}(H -$

$1) \cap \mathbf{B}(H)]$ gives a vector which is still in the group, moreover according to MR this operation does not affect the evaluation of the vector, i.e. $\phi(\mathbf{b}') = \phi(\mathbf{b})$ for all $\mathbf{b}, \mathbf{b}' \in \mathbf{B}(H-1) \backslash [\mathbf{B}(H-1) \cap \mathbf{B}(H)]$. If $H \geq 3$, next equivalence class is obtained considering all vectors in $\mathbf{B}(H-2) \backslash [\mathbf{B}(H-2) \cap [\mathbf{B}(H) \cup \mathbf{B}(H-1)]$. The sequence is completed with the last element $\mathbf{B}(1) \backslash \mathbf{B}(1) \cap [\cup_{j=2}^{H} \mathbf{B}(j)]$ composed by all vectors $\mathbf{1}_n$ for $n \geq 1$.

Note that all these equivalence classes can be associated with the value of $\mathbf{b}^{\max}$ which is the same for all the elements in a class. Thus we let $\phi(\mathbf{b}) = \psi(\mathbf{b}^{\max})$ where ψ is non-decreasing. We have obtained (3).

According to N we have that $v(1) \cdot \psi(1) = 0$, and $v(H) \cdot \psi(H) = 1$.

To complete note that MR is consistent with (4), $w(b_i; \mathbf{b}) = 0$ for all $\mathbf{b} \in \mathbf{B}(b_i)$.

The *sufficiency part* of the proof requires to check that (3) satisfies A, WM, WMA, SRI, N, and MR. ■

Proof of Remark 1. Axiom IEF requires that if $v(b_i) \cdot \psi(\mathbf{b}^{\max}) \geq v(b_i) \cdot \psi(\mathbf{b'}^{\max})$ then $v(b_i) \cdot \psi(\max\{\mathbf{b}^{\max}, b_0\}) \geq v(b_i) \cdot \psi(\max\{\mathbf{b'}^{\max}, b_0\})$ for all $b_0 \in N_H$. To check that IEF is satisfied then we consider that,

(i)either $v(b_i) = 0$, thus any evaluation of the achievement b_i is equal to 0, and IEF is satisfied,

(ii) or $v(b_i) > 0$, thus $\psi(\mathbf{b}^{\max}) \geq \psi(\mathbf{b'}^{\max})$ should imply $\psi(\max\{\mathbf{b}^{\max}, b_0\}) \geq \psi(\max\{\mathbf{b'}^{\max}, b_0\})$.

(iia) If $\mathbf{b}^{\max} \geq \mathbf{b'}^{\max}$ then $\max\{\mathbf{b}^{\max}, b_0\} \geq \max\{\mathbf{b'}^{\max}, b_0\}$ thus, since ψ is non-decreasing then IEF is satisfied.

(iib) If $\mathbf{b}^{\max} < \mathbf{b'}^{\max}$ and $\psi(\mathbf{b}^{\max}) = \psi(\mathbf{b'}^{\max})$ then (1) either $b_0 \leq \mathbf{b}^{\max}$ implying that $\max\{\mathbf{b}^{\max}, b_0\} = \mathbf{b}^{\max} < \mathbf{b'}^{\max} = \max\{\mathbf{b'}^{\max}, b_0\}$; or (2) $\mathbf{b}^{\max} \leq b_0 \leq \mathbf{b'}^{\max}$ implying that $\max\{\mathbf{b'}^{\max}, b_0\} = b_0 \leq \mathbf{b'}^{\max} = \max\{\mathbf{b'}^{\max}, b_0\}$ which according to WM and $\psi(\mathbf{b}^{\max}) = \psi(\mathbf{b'}^{\max})$ requires that $\psi(b_0) = \psi(\mathbf{b'}^{\max})$; or finally (3) $\mathbf{b}^{\max} < \mathbf{b'}^{\max} < b_0$ which implies that $\max\{\mathbf{b'}^{\max}, b_0\} = b_0 = \max\{\mathbf{b'}^{\max}, b_0\}$ thereby completing the proof that IEF is satisfied. ■

Proof of Proposition 2. *Necessity:* Consider the result of Lemma 1, axiom TR requires that $\phi(\mathbf{b}') \geq \phi(\mathbf{b})$ if $\mathbf{b}' = \mathbf{b} \cup b^*$ and $b^* > z$, otherwise $\phi(\mathbf{b}') = \phi(\mathbf{b})$ if $b^* \leq z$.

Note that applying TR (part ii) we obtain that if $\beta \leq z$ then $\phi(\mathbf{b}') = \phi(\mathbf{b}) = \phi(z \cdot \mathbf{1}_n) = \phi(z)$ for all $\mathbf{b}, \mathbf{b}' \in \mathbf{B}_n$ such that $\mathbf{b}, \mathbf{b}' \leq z \cdot \mathbf{1}_n$, where the last equality is obtained by applying RI.

In general, making use of (4) we get $\phi(\mathbf{b}') = \phi(\mathbf{b})$ if $\boldsymbol{\alpha}(\mathbf{b}, z) = \boldsymbol{\alpha}(\mathbf{b}', z)$. Therefore $\boldsymbol{\alpha}(\mathbf{b}, z)$ should be considered in order to identify distributions exhibiting the same value of ϕ. Note that according to TR (part i) $\phi(\mathbf{b}') = \phi(\mathbf{b})$ for any pair of vectors $\mathbf{b}' \in \mathbf{B}_{n'}, \mathbf{b} \in \mathbf{B}_n$ such that $n' \geq n$, if $\boldsymbol{\alpha}(\mathbf{b}', z) = \boldsymbol{\alpha}(\mathbf{b}, z) \cup z\mathbf{1}_{n'-n}$. By applying RI we get that $\phi(\mathbf{b}') = \phi(\mathbf{b})$ for all $\mathbf{b}' \in \mathbf{B}_{n'}, \mathbf{b} \in \mathbf{B}_n$ if either $\boldsymbol{\alpha}(\mathbf{b}', z) = z\mathbf{1}_{n'}$ and $\boldsymbol{\alpha}(\mathbf{b}, z) = z\mathbf{1}_n$ or $\mathbf{b}'$ and $\mathbf{b}$ exhibit the same distribution of functioning above z. That is the valuation is made according to the distribution of functioning in $\boldsymbol{\alpha}^{T*}(\mathbf{b}, z)$.

Recalling that WM requires that $\phi(\mathbf{b})$ is non-decreasing in each argument, then the evaluation $\phi(\mathbf{b}) := \zeta(\boldsymbol{\alpha}^{T*}(\mathbf{b}, z))$ should be non-decreasing in each component of $\boldsymbol{\alpha}^{T*}(\mathbf{b}, z)$.

According to N we have that $v(1) \cdot \zeta(z) = 0$, and $v(H) \cdot \zeta(H) = 1$.

To complete note that TR is consistent with $w(b_i; \mathbf{b}) = 0$ for all $\mathbf{b} \in \mathbf{B}(b_i)$.

The *sufficiency part* of the proof requires to check that (6) satisfies A, WM, WMA, SRI, N, and TR. ■

Proof of Remark 3. Recalling definition of Axiom IEF then if $b_0 \leq z$ then $\boldsymbol{\alpha}^{T*}(\mathbf{b}, z)$ is not affected and similarly is (6) thus IEF is satisfied. However if $b_0 > z$ then IEF requires that $\zeta[\boldsymbol{\alpha}^{T*}(\mathbf{b} \cup b_0, z)] \geq \zeta[\boldsymbol{\alpha}^{T*}(\mathbf{b}' \cup b_0, z)]$ if $\zeta[\boldsymbol{\alpha}^{T*}(\mathbf{b}, z)] \geq \zeta[\boldsymbol{\alpha}^{T*}(\mathbf{b}', z)]$ which may not be the case. For instance consider the following specification for ζ where $\boldsymbol{\alpha}^{T*}(\mathbf{b}, z)^{\max}$ and $\boldsymbol{\alpha}^{T*}(\mathbf{b}, z)_{\min}$ respectively denote the maximum and minimum value of $\boldsymbol{\alpha}^{T*}(\mathbf{b}, z)$:

$$\zeta_{\min}^{\max}[\boldsymbol{\alpha}^{T*}(\mathbf{b}, z)] = \frac{1}{2}[\boldsymbol{\alpha}^{T*}(\mathbf{b}, z)^{\max} + \boldsymbol{\alpha}^{T*}(\mathbf{b}, z)_{\min}].$$

Note that $\zeta_{\min}^{\max}$ is replication invariant, symmetric and non-decreasing in each element. Suppose that $z = 2$, $\boldsymbol{\alpha}^{T*}(\mathbf{b}, 2) = (3, 5, 5, 7)$ and $\boldsymbol{\alpha}^{T*}(\mathbf{b}', 2) = (4, 4, 4, 5)$, while $b_0 = H = 7$ then $\zeta_{\min}^{\max}\left[\boldsymbol{\alpha}^{T*}(\mathbf{b}, z)\right] = \frac{1}{2}(7{+}3) > \frac{1}{2}(5{+}4) = \zeta_{\min}^{\max}[\boldsymbol{\alpha}^{T*}(\mathbf{b}', 2)]$, while since $\boldsymbol{\alpha}^{T*}(\mathbf{b} \cup 7, 2) = (3, 5, 5, 7, 7)$ and $\boldsymbol{\alpha}^{T*}(\mathbf{b}' \cup 7, 2) = (4, 4, 4, 5, 7)$ we have $\zeta_{\min}^{\max}[\boldsymbol{\alpha}^{T*}(\mathbf{b} \cup 7, 2)] = \frac{1}{2}(7 + 3) < \frac{1}{2}(7 + 4) = \zeta_{\min}^{\max}[\boldsymbol{\alpha}^{T*}(\mathbf{b}' \cup 7, 2)]$ thereby violating IEF.

Another example can be presented taking ζ to coincide with the averaging operator μ giving the mean value of a vector and letting $\boldsymbol{\alpha}^{T*}(\mathbf{b}', 2) = (4)$, as a result comparing averages we get $\mu(3, 5, 5, 7) = 5 > 4 = \mu(4)$ while adding $b_0 = 7$ we get $\mu(3, 5, 5, 7, 7) = 5.4 < 5.5 = \mu(4, 7)$, thereby violating IEF. ■

Proof of Proposition 3. *Necessity:* Consider the result of Lemma 1, axiom TC (part i) combined with A requires that $\phi(\mathbf{b}') = \phi(\mathbf{b})$ for any $\beta \in N_H$ whenever $\mathbf{b}', \mathbf{b} \in \mathbf{B}_n(\beta)$ are obtained changing functionings levels and relative positions of individuals without affecting the ranking of those experiencing functioning z. Applying RI the above consideration extends to all cases where $\bar{F}_{\mathbf{b}}(z)$ is not affected. That is, let $\beta \in N_H$ and $\mathbf{b}', \mathbf{b} \in \mathbf{B}(\beta)$ then $\phi(\mathbf{b}') = \phi(\mathbf{b})$ whenever $\bar{F}_{\mathbf{b}}(z) = \bar{F}_{\mathbf{b}'}(z)$.

In order to extend the result to all $\mathbf{b}', \mathbf{b} \in \mathbf{B}$, consider $\mathbf{b}', \mathbf{b} \in \mathbf{B}(\beta, \beta')$, then axiom TC (part i) in conjunction with A and RI requires that if $\bar{F}_{\mathbf{b}}(z) = \bar{F}_{\mathbf{b}'}(z)$ then $\phi(\mathbf{b}') = \phi(\mathbf{b})$ both if $b_j = \beta$ and if $b_j = \beta'$ provided that $v(\beta) > 0$ and $v(\beta') > 0$. Thus $\bar{F}_{\mathbf{b}}(z) = \bar{F}_{\mathbf{b}'}(z)$ will imply that $\phi(\mathbf{b}') = \phi(\mathbf{b})$ for any $\mathbf{b}', \mathbf{b} \in \mathbf{B}(\beta) \cup \mathbf{B}(\beta')$. Repeated application of the procedure extends the result to hold for all $\mathbf{b}', \mathbf{b} \in \mathbf{B}$.

Recall that if $v(b_i) = 0$ then $w(b_i; \mathbf{b}) = 0$ consistently with the specification in (7), thus the only relevant case to check is when $v(b_i) > 0$.

If $v(b_i) > 0$, we have shown that $\bar{F}_{\mathbf{b}}(z) = k \in \mathbb{Z}_{[0,1]}$ identifies an equivalence class for all $\mathbf{b} \in \mathbf{B}$. According to axiom TC (part ii) if $\mathbf{b} = \mathbf{b}^* \cup \beta \in \mathbf{B}_n$ where $\beta \leq z$, and $\mathbf{b}' = \mathbf{b}^* \cup \beta' \in \mathbf{B}_n$ where $\beta' > z$ then $\phi(\mathbf{b}') \geq \phi(\mathbf{b})$. Note that

$\bar{F}_{\mathbf{b}}(z) = k < k + 1/n = \bar{F}_{\mathbf{b}'}(z)$. Moreover note that for any $k, k' \in \mathbb{Z}_{[0,1]}$ s.t. $k < k'$ there exist $\rho, \rho', n \in \mathbb{N}$ such that $k = \rho/n$, $k' = \rho'/n$.

Let $z \in N_H$. Consider $\mathbf{b}', \mathbf{b} \in \mathbf{B}$ s.t. $\bar{F}_{\mathbf{b}}(z) = k < k' = \bar{F}_{\mathbf{b}'}(z)$. Then there exist $\mathbf{b}'_1, \mathbf{b}_1 \in \mathbf{B}_n$ s.t. $k = \rho/n$, $k' = \rho'/n$ and $\bar{F}_{\mathbf{b}_1}(z) = k < k' = \bar{F}_{\mathbf{b}'_1}(z)$. Let

$$\begin{aligned}\hat{\mathbf{b}}_1 \quad &: \quad = (\underbrace{z, z, z, ...z, z}_{\rho}, \underbrace{H, ...H}_{n-\rho}),\\ \hat{\mathbf{b}}'_1 \quad &: \quad = (\underbrace{z, .., z, z}_{\rho'}, \underbrace{H, H, H, ...H}_{n-\rho'}),\end{aligned}$$

thus $\hat{\mathbf{b}}_1, \hat{\mathbf{b}}'_1$ are distributions of size n, ranked in non-decreasing order (obtained re-ranking respectively $\mathbf{b}'$ and $\mathbf{b}$) whose first elements are $z's$ and the other elements of higher value are $H's$.

Consider two cases, either (I) $z < H$, or (II) $z = H$.

(I) If $z < H$ consider two generic distributions $\mathbf{b}', \mathbf{b} \in \mathbf{B}$, then

(I.a) as shown above, if $\bar{F}_{\mathbf{b}'}(z) = \bar{F}_{\mathbf{b}'_0}(z)$ then by TC (part i) $\phi(\mathbf{b}') = \phi(\mathbf{b}'_0)$ and if $\bar{F}_{\mathbf{b}}(z) = \bar{F}_{\mathbf{b}_0}(z)$ then by TC (part i) $\phi(\mathbf{b}) = \phi(\mathbf{b}_0)$,

(I.b) by RI $\phi(\mathbf{b}'_0) = \phi(\mathbf{b}'_1)$ and $\phi(\mathbf{b}_0) = \phi(\mathbf{b}_1)$,

(I.c) by A $\phi(\mathbf{b}'_1) = \phi(\hat{\mathbf{b}}'_1)$ and $\phi(\mathbf{b}_1) = \phi(\hat{\mathbf{b}}_1)$,

(I.d) by TC (part ii), repeated applications of comparisons where starting from $\hat{\mathbf{b}}_1$ another distribution in considered where a functioning of value z is replaced by another of value H, we obtain $\phi(\hat{\mathbf{b}}'_1) \geq \phi(\hat{\mathbf{b}}_1)$.

Collecting all results and noticing that $\bar{F}_{\mathbf{b}'}(z) = \bar{F}_{\hat{\mathbf{b}}'_1}(z) > \bar{F}_{\hat{\mathbf{b}}_1}(z) = \bar{F}_{\mathbf{b}}(z)$ we obtain that

$$\phi(\mathbf{b}') = \phi(\mathbf{b}'_0) = \phi(\mathbf{b}'_1) = \phi(\hat{\mathbf{b}}'_1) \geq \phi(\hat{\mathbf{b}}_1) = \phi(\mathbf{b}_1) = \phi(\mathbf{b}_0) = \phi(\mathbf{b}).$$

Thus, if $\bar{F}_{\mathbf{b}'}(z) > \bar{F}_{\mathbf{b}}(z)$ then $\phi(\mathbf{b}') \geq \phi(\mathbf{b})$.

(II) If $z = H$ then by definition $\bar{F}_{\mathbf{b}}(z) = \bar{F}_{\mathbf{b}'}(z) = 0$ for all $\mathbf{b}', \mathbf{b} \in \mathbf{B}$. As shown above $\bar{F}_{\mathbf{b}}(z) = \bar{F}_{\mathbf{b}'}(z)$ will imply that $\phi(\mathbf{b}') = \phi(\mathbf{b})$ for all $\mathbf{b}', \mathbf{b} \in \mathbf{B}$. Thus if $z = H$ then $\phi(\mathbf{b}') = \phi(\mathbf{b})$ for all $\mathbf{b}', \mathbf{b} \in \mathbf{B}$.

To summarize, for a given $z \in N_H$ function $\phi(\mathbf{b})$ can be specified in terms of $\bar{F}_{\mathbf{b}}(z)$, that is $\phi(\mathbf{b}) := \chi_z(\bar{F}_{\mathbf{b}}(z))$ where χ_z may depend on the value of z and is non-decreasing in $\bar{F}_{\mathbf{b}}(z)$. Note that so fare we didn't make use of WM which turns out to be implied by the result.

According to N we have that $v(1) \cdot \chi_z(0) = 0$, implying $v(1) = 0$ since by construction $\chi_z(.) > 0$. Moreover N requires that $v(H) \cdot \chi_z(1) = 1$ if $z < H$, and if $z = H$ we have $v(H) \cdot \chi_H(0) = 1$.

To complete note that TR is consistent with $w(b_i; \mathbf{b}) = 0$ for all $\mathbf{b} \in \mathbf{B}(b_i)$.

The *sufficiency part* of the proof requires to check that (7) satisfies A, WM, WMA, SRI, N, and TC. ■

Proof of Remark 4. Recalling definition of Axiom IEF then (7) satisfies IEF if $\chi(z, \bar{F}_{\mathbf{b}\cup b_0}(z)) \geq \chi(z, \bar{F}_{\mathbf{b}'\cup b_0}(z))$ holds if condition $\chi(z, \bar{F}_{\mathbf{b}}(z)) \geq \chi(z, \bar{F}_{\mathbf{b}'}(z))$ is satisfied. Thus for a fixed z IEF requires that $\bar{F}_{\mathbf{b}\cup b_0}(z) \geq \bar{F}_{\mathbf{b}'\cup b_0}(z)$ if $\bar{F}_{\mathbf{b}}(z) \geq$

$\bar{F}_{\mathbf{b}'}(z)$. Let $n(\mathbf{b})$ denote the size of distribution $\mathbf{b}$, suppose $b_0 \leq z$, $n(\mathbf{b}) = 10$, $n(\mathbf{b}') = 100$ and $\bar{F}_{\mathbf{b}}(z) = 4/10 > 39/100 = \bar{F}_{\mathbf{b}'}(z)$. However once we consider distribution $\mathbf{b} \cup b_0$ and $\mathbf{b}' \cup b_0$ we get $\bar{F}_{\mathbf{b} \cup b_0}(z) = 4/11 < 39/101 = \bar{F}_{\mathbf{b}' \cup b_0}(z)$ thereby violating IEF. ■

Proof of Corollary 2. Recalling (7) then for a given $z \in N_H$ the group evaluation is non-decreasing with $\bar{F}_{\mathbf{b}}(z)$.

While according to (6) the group evaluation depends on $\boldsymbol{\alpha}^T(\mathbf{b}, z)$ if at least one functioning in $\mathbf{b}$ is above z i.e. if $\bar{F}_{\mathbf{b}}(z) > 0$, otherwise (i.e. if $\bar{F}_{\mathbf{b}}(z) = 0$) $\boldsymbol{\alpha}^T(\mathbf{b}, z)$ does not exist and the group evaluation reaches a minimum.

Note that for a given $\bar{F}_{\mathbf{b}}(z) > 0$ then combining the two criteria the composition of $\boldsymbol{\alpha}^T(\mathbf{b}, z)$ is irrelevant for the evaluation since what matters is only $\bar{F}_{\mathbf{b}}(z)$. Moreover according to (6) the group evaluation has to satisfy RI when applied also on $\boldsymbol{\alpha}^T(\mathbf{b}, z)$. By replicating $\boldsymbol{\alpha}^T(\mathbf{b}, z)$ keeping unchanged the elements of $\mathbf{b}$ not above the threshold z the value of $\bar{F}_{\mathbf{b}}(z)$ is increased. While by replicating all elements not above z i.e. $\mathbf{b} \backslash \boldsymbol{\alpha}^T(\mathbf{b}, z)$ while keeping fixed $\boldsymbol{\alpha}^T(\mathbf{b}, z)$ then the value of $\bar{F}_{\mathbf{b}}(z)$ is decreased. Note that both operations do not affect the evaluation in (6).

It can be shown that for any $k \in \mathbb{Z}_{(0,1]}$, for a given $\bar{F}_{\mathbf{b}}(z) > 0$ it is always possible to replicate distribution $\mathbf{b} \backslash \boldsymbol{\alpha}^T(\mathbf{b}, z)$ and replicate a different number of time distribution $\boldsymbol{\alpha}^T(\mathbf{b}, z)$ s.t. for the obtained distribution $\mathbf{b}'$ we have $\bar{F}_{\mathbf{b}'}(z) = k$.

Thus the only discriminant information is whether $\bar{F}_{\mathbf{b}}(z) > 0$ or $\bar{F}_{\mathbf{b}}(z) = 0$.

■

Proof of Lemma 2. In order to simplify the exposition we consider directly a FCRF where we let $\mathbf{b}^{\max} := m$ and $w(b_i; \mathbf{b})$ is represented by $\tilde{w}(b; m) = v(b) \cdot \psi(m)$. Property WIP requires that:

$$v(b) \cdot \psi(m) \geq v(b-1) \cdot \psi(m') \tag{29}$$

for all $b, m, m' \in N_H$ such that $1 < b \leq m \leq H$, $H \geq m' \geq b - 1 \geq 1$, $m' \in \{m-1, m, m+1\}$.

The non trivial case for these comparisons arises when $m' = m+1$, given the assumptions on v and ψ the condition (29) is always satisfied if $m' \in \{m-1, m\}$. For $m' = m + 1$ we have

$$v(b) \cdot \psi(m) \geq v(b-1) \cdot \psi(m+1)$$

which is satisfied either if $v(b-1) = 0$, or if $v(b-1) > 0$ and

$$\frac{v(b)}{v(b-1)} \geq \frac{\psi(m+1)}{\psi(m)} \text{ for all } m \geq b > 2.$$

Therefore letting $\underline{b} := \max\{b \in N_H : v(b) = 0\}$ we obtain

$$\min_{m \geq b > \underline{b}+1} \left\{ \frac{v(b)}{v(b-1)} \right\} \geq \frac{\psi(m+1)}{\psi(m)} \text{ for any } m \leq H - 1. \tag{30}$$

■

Proof of Proposition 4. For expositional purposes we split the proof into three parts.

We first consider the functions $v_{\{\underline{b},\bar{b}\}}$ as in (21) for fixed $1 \le \underline{b} < \bar{b} \le H$ and derive the specification of the set $\Psi(\underline{b},\bar{b})$ of all consistent values of ψ satisfying WIP following Lemma (2) (see Part A).

Then we derive [see (34)] a parametric function $\psi^*_{\{\underline{b},\bar{b}\}} \in \Psi(\underline{b},\bar{b})$ that will turn out to be essential for characterizing together with $v_{\{\underline{b},\bar{b}\}}$ two conditions that guarantee robust cross group comparisons (see Part B).

We will complete the proof deriving consistently with the results in Part B the dominance conditions specified directly comparing individual achievements b and maximal group realizations m.

(Part A): We first specify the implications arising from applying the result in Lemma 2 when v is specified making use of $v_{\{\underline{b},\bar{b}\}}(b)$.

Consider (21), note that since $v(H) = 1$ we get also $\psi(H) = 1/v(H) = 1$.Then if $\bar{b} < H$ we get $\psi(b) = 1$ for all $b \in \{\bar{b}+1, \bar{b}+2, ..H\}$.

For all $b \in \{\underline{b}+2, \underline{b}+3, ...\bar{b}\}$ we have

$$\frac{v_{\{\underline{b},\bar{b}\}}(b)}{v_{\{\underline{b},\bar{b}\}}(b-1)} = \frac{b-\underline{b}}{b-1-\underline{b}} = 1 + \frac{1}{b-1-\underline{b}}$$

thus according to (20)

$$\min_{m \ge b \ge \underline{b}+2} \left\{ 1 + \frac{1}{b-1-\underline{b}} \right\} = 1 + \frac{1}{m-1-\underline{b}} \ge \frac{\psi(m+1)}{\psi(m)} \quad \text{for any } m \in \{\underline{b}+2, \underline{b}+3, ...\bar{b}\}. \tag{31}$$

While for $b \in \{1, 2, .., \underline{b}, \underline{b}+1\}$ we get no restrictions on $\psi(b)$. However, since $v(b) = 0$ for all $b \in \{1, 2, .., \underline{b}\}$ and the FCRF in (3) is $\tilde{w}(b,m) = v(b) \cdot \psi(m)$ then the values of $\psi(b)$ for all $b \in \{1, 2, .., \underline{b}\}$ are of *no relevance* for the final evaluation. Possibly the value of $\psi(\underline{b}+1)$ may turn out to be relevant, but this is not the case. The only possible evaluation for which $\psi(\underline{b}+1)$ can be relevant is the one considering $\tilde{w}(\underline{b}+1, \underline{b}+1) = v(\underline{b}+1) \cdot \psi(\underline{b}+1)$, non trivial between groups comparisons involve individuals with $\tilde{w}(b,m)$ where $b < \underline{b}+1$ and $m > \underline{b}+1$ but these individuals experience achievements $b \le \underline{b}$ therefore for them we have $\tilde{w}(b,m) = 0$. Thus the value of $\psi(\underline{b}+1) > 0$ does not affect any between group comparison.

(Part B): We now move to identify an algorithm to implement cross group comparisons in terms of FCRFs. We consider the set of FCRF $\tilde{w}(b,m) = v(b) \cdot \psi(m)$ derived in Proposition 1 where m denotes the maximal group achievement, and b is the level of individual achievement. We make use of the result in Part A and denote by $\Psi(\underline{b},\bar{b})$ the set of all non-decreasing functions $\psi > 0$ s.t. $\psi(H) = 1$ and $\tilde{w}(b,m)$ satisfies WIP for $v_{\{\underline{b},\bar{b}\}}(b)$ given in (21).

We are interested in robust cross group comparisons holding for a given $v_{\{\underline{b},\bar{b}\}}$ and for all associated ψ belonging to .

Any between group nontrivial comparison in terms of FCRFs requires to rank $\tilde{w}(b,m) = v(b)\cdot\psi(m) \geq v(b')\cdot\psi(m') = \tilde{w}(b',m')$ in particular when $b' < b \leq m < m'$.

Letting $\psi(m) = 1$ for all $m \in N_H$ it follows that $\tilde{w}(b,m) \geq \tilde{w}(b',m')$ if and only if $v(b) \geq v(b')$.

In order to check whether the same relation holds also for other admissible specifications of $\psi \in \Psi(\underline{b},\bar{b})$ the condition

$$v(b) \geq v(b') \max_{\psi\in\Psi(\underline{b},\bar{b})} \left\{\frac{\psi(m')}{\psi(m)}\right\} \tag{32}$$

has to hold.

Since $\psi(m')/\psi(m)$ can be decomposed into the product of ratios of $\psi(j+1)/\psi(j)$ for $j = m, ...m'-1$ the maximum value of $\psi(m')/\psi(m)$ necessarily has to coincide with the product of the maximum values of each ratio $\psi(j+1)/\psi(j)$ i.e.

$$\max_{\psi\in\Psi(\underline{b},\bar{b})} \left\{\frac{\psi(m')}{\psi(m)}\right\} = \prod_{j=m}^{m'-1} \max_{\psi\in\Psi(\underline{b},\bar{b})} \left\{\frac{\psi(j+1)}{\psi(j)}\right\}. \tag{33}$$

From Part A we know that $\psi(j) = 1$ for all $j \in \{\bar{b}+1, \bar{b}+2, ..H\}$ and from (31) we have that $1 + \frac{1}{j-1-\underline{b}} \geq \frac{\psi(j+1)}{\psi(j)}$ for any $j \in \{\underline{b}+2, \underline{b}+3, ...\bar{b}\}$, therefore

$$\max_{\psi\in\Psi(\underline{b},\bar{b})} \left\{\frac{\psi(j+1)}{\psi(j)}\right\} = \frac{j-\underline{b}}{j-1-\underline{b}}.$$

Our aim is to identify a possibly unique function $\psi^* \in \Psi(\underline{b},\bar{b})$ that can be considered as the *"limiting function"* among those in $\Psi(\underline{b},\bar{b})$ such that $v(b)\cdot\psi^*(m) \geq v(b')\cdot\psi^*(m')$ will imply that $v(b)\cdot\psi(m) \geq v(b')\cdot\psi(m')$ for all $\psi \in \Psi(\underline{b},\bar{b})$.

Because of (32), (**??**) and (31) the restrictions on ψ^* require that:

$$\frac{\psi^*(b')}{\psi^*(b)} = \max_{\psi\in\Psi(\underline{b},\bar{b})} \left\{\frac{\psi(b')}{\psi(b)}\right\} \text{ for all } \underline{b}+2 \leq b < b' \leq H.$$

It follows that $\psi^*(b) = 1$ for all $b \in \{\bar{b}+1, \bar{b}+2, ..H\}$ and $\psi^*(b) = \theta\cdot(b-1-\underline{b})$ for any $b \in \{\underline{b}+2, \underline{b}+3, ...\bar{b}\}$ where θ is a constant with appropriate value s.t. according to (20)

$$\frac{v_{\{\underline{b},\bar{b}\}}(\bar{b})}{v_{\{\underline{b},\bar{b}\}}(\bar{b}-1)} = \frac{\psi^*(\bar{b}+1)}{\psi^*(\bar{b})}.$$

Since applying (21) we have that $v_{\{\underline{b},\bar{b}\}}(\bar{b}) = 1 = \psi^*(\bar{b}+1)$, it follows that $\psi^*(\bar{b}) = v_{\{\underline{b},\bar{b}\}}(\bar{b}-1)$ i.e. $\theta = 1/(\bar{b}-\underline{b})$, thus

$$\psi^*(b) = \frac{b-1-\underline{b}}{\bar{b}-\underline{b}} = v_{\{\underline{b},\bar{b}\}}(b-1)$$

for all $b \in \{\underline{b}+2, \underline{b}+3, ...\bar{b}\}$.

The set of conditions previously derived allows to recover a unique derivation of $\psi^*(b)$ for $b \geq \underline{b}+2$, however for $b \leq \underline{b}+1$ the function ψ^* may exhibit different shapes provided that it is positive and non-decreasing. As argued in Part A the values of $\psi^*(b)$ for $b \in \{1, 2, ...\underline{b}, \underline{b}+1\}$ have no relevance in between groups comparisons since $v_{\{\underline{b},\bar{b}\}}(b) = 0$ for all $b \in \{1, 2, ...\underline{b}\}$ and even if $v_{\{\underline{b},\bar{b}\}}(\underline{b}+1) \cdot \psi^*(\underline{b}+1) > 0$ then either

(i) $b' < \underline{b}+1$ i.e. $v_{\{\underline{b},\bar{b}\}}(b') = 0$, in which case $v_{\{\underline{b},\bar{b}\}}(\underline{b}+1) \cdot \psi^*(\underline{b}+1) > v_{\{\underline{b},\bar{b}\}}(b') \cdot \psi^*(m')$ no matter the value of $\psi^*(\underline{b}+1)$ and the value of $m' \geq b'$, or

(ii) $b' \geq \underline{b}+1$, which is necessarily associated with $m' \geq \underline{b}+1$ clearly giving either $v_{\{\underline{b},\bar{b}\}}(\underline{b}+1) \cdot \psi^*(\underline{b}+1) = v_{\{\underline{b},\bar{b}\}}(b') \cdot \psi^*(m')$ if $b' = \underline{b}+1 = m'$ or $v_{\{\underline{b},\bar{b}\}}(\underline{b}+1) \cdot \psi^*(\underline{b}+1) \leq v_{\{\underline{b},\bar{b}\}}(b') \cdot \psi^*(m')$ otherwise, no matter what is the value of $\psi^*(\underline{b}+1)$.

As a result an admissible specification for ψ^* is $\psi^*(b) := v_{\{\underline{b},\bar{b}\}}(b-1)$ for $b \in \{\underline{b}+2, \underline{b}+3, .., \bar{b}, ..H\}$ and $\psi^*(b) := v_{\{\underline{b},\bar{b}\}}(\underline{b}+1)$ for $b \in \{1, 2, ..\underline{b}+1\}$, i.e.

$$\psi^*_{\{\underline{b},\bar{b}\}}(b) = \begin{cases} \frac{1}{\bar{b}-\underline{b}} & \text{if } 1 \leq b \leq \underline{b}+1 \\ \frac{b-1-\underline{b}}{\bar{b}-\underline{b}} & \text{if } \underline{b}+1 < b \leq \bar{b}+1 \\ 1 & \text{if } \bar{b}+1 < b \leq H \end{cases} . \tag{34}$$

Robust comparisons of two individuals belonging to different groups in terms of the FCRFs derived in Proposition 1 once $v_{\{\underline{b},\bar{b}\}}(b)$ in (21) is applied, require to check whether $\tilde{w}(b,m) = v(b) \cdot \psi(m) \geq v(b') \cdot \psi(m') = \tilde{w}(b',m')$ for $v_{\{\underline{b},\bar{b}\}}$ and for all $\psi \in \Psi(\underline{b},\bar{b})$. As shown in the previous paragraphs in order to implement these comparisons, if $v_{\{\underline{b},\bar{b}\}}(b) \geq v_{\{\underline{b},\bar{b}\}}(b')$ then the specification of ψ in (34) it is not only necessary but *it is also sufficient.* To summarize:

For a fixed pair of values $\{\underline{b},\bar{b}\}$ we have that $v_{\{\underline{b},\bar{b}\}}(b) \cdot \psi(m) \geq v_{\{\underline{b},\bar{b}\}}(b') \cdot \psi(m')$ for all $\psi \in \Psi(\underline{b},\bar{b})$ if and only if:
(i) $v_{\{\underline{b},\bar{b}\}}(b) \geq v_{\{\underline{b},\bar{b}\}}(b')$, and
(ii) $v_{\{\underline{b},\bar{b}\}}(b) \cdot \psi^*_{\{\underline{b},\bar{b}\}}(m) \geq v_{\{\underline{b},\bar{b}\}}(b') \cdot \psi^*_{\{\underline{b},\bar{b}\}}(m')$.

Part (C): Our aim is to restate these conditions directly in terms of the values of b, m, b', m'.

Condition (i) requires that either (i.a) $b \geq b'$ or (i.b) $b, b' \leq \underline{b}$ or (i.c) $b, b' \geq \bar{b}$.

Condition (ii) turns out to be trivial either when $m \geq m'$ or $b = b'$, and in general if either (i.b) or (i.c) hold given that if (i.b) holds we have that the value of the two FCRFs in (ii) is 0 and if (i.c) hold we obtain that the value of the two FCRFs in (ii) is 1.

When $b' < b \leq m < m'$ we have a non-trivial comparison. In this case condition (ii) requires that:

$$\max\left\{\frac{b-\underline{b}}{\bar{b}-\underline{b}}; 0\right\} \cdot \min\left\{\frac{[m-1-\underline{b}]_{[1]}}{\bar{b}-\underline{b}}; 1\right\}$$
$$\geq \max\left\{\frac{b'-\underline{b}}{\bar{b}-\underline{b}}; 0\right\} \cdot \min\left\{\frac{[m'-1-\underline{b}]_{[1]}}{\bar{b}-\underline{b}}; 1\right\}$$

that is after normalization

$$
\begin{aligned}
& \max\left\{b-\underline{b};0\right\}\cdot\min\left\{\left[m-1-\underline{b}\right]_{[1]};\bar{b}-\underline{b}\right\} \\
\geq\quad & \max\left\{b'-\underline{b};0\right\}\cdot\min\left\{\left[m'-1-\underline{b}\right]_{[1]};\bar{b}-\underline{b}\right\}.
\end{aligned}
\tag{35}
$$

■