# ON MEASURING DIVERSITY

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## - PRELIMINARY VERSION -

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Abstract. In the last ten years, the measurement of diversity has became an increasingly important issue in the literature. In this work we propose a diversity index and an ordering based on the couting approach. We present an axiomatic characterization of the index and the ordering.

"*The individuals may differ to characteristics that are deemed relevant for the evaluation of income distribution*" (Serge-Christophe Kolm)[1]

## 1. Introduction

Diversity is a recurring concept. We use the word *diversity* related to languages, religions, cultural, countries, people and so on.

But is diversity good or bad? In general, in a society or in a social setting, diversity would be desirable. In fact people or groups would obtain benefit from having different skills, different talents, different options or different points of view. Sometimes we are interesting in eliminating diversity for example in a context of integration. At times we are interesting in preserving diversity for instance, as suggested by Bossert, Pattanaik and Xu [7], in the case of political options for the voters in a country because, in this context, having diversity means have more options to choose and then, more freedom.

The definition and measurement of diversity are two topics of interest to many scholars.

In order to measure diversity, the first scholars interesting in developing indices have been Biologists. They translated the entropy measure proposed by Shannon [17] into the biology field, constructing a class of indices. For a given ecosystem, this class evaluates the diversity first, by counting, for each species, the frequency of living individuals within the species relative to the total number of living individual and then, by calculating a weighted entropy over these relative frequencies. One limitation in using Shannon's entropy is that these measures do not take in account the inter-species diversity or if two individuals of the same species are more diverse than two individuals belonging to two different species.

In the economics literature, many scholars have made important contributions to measurement of diversity. Weitzman [19] derives the diversity of a set from pairwise dissimilarity between its elements. His work is based on the primitive notion of a cardinal numeric measure of distance between living creature. Nehring and Puppe [12] generalize this measure; in their work, they propose to derive the Weitzman's distance functions from an a prior grouping of the objects into collections of weighted[2] attributes. Afterwords, Bossert, Pattanaik and Xu [7] propose an axiomatic characterization of the Weitzman's criterion by proposing a leximin elimination criterion with respect to the minimal distance

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[1] *Rational fundatuin of inequality measurement*, Handbook on income inequality measurement, p 81.

[2] These weight derive from a numerical function based on cardinality.

between an object in a set and the remaining elements of the set. After these pioneer works, many researchers have been interested in measuring diversity based on an ordinal distance. Pattanaik and Xu [14] provide and improve a distance function that establishes if objects are similar or dissimilar and measures the diversity of different sets. Bervoets and Gravel [4] give an axiomatic characterization of two different ways of ranking sets on the basis of the diversity that they offer. They assume that a set is more diverse that another if and only if the two most dissimilar objects in the former are weakly as dissimilar as the two most dissimilar objects in the later

Finally, Pattanaik and Xu [15] suggest an ordinal distance function to develop a notion of dominance between sets and to characterize a specific ranking rule belonging to a class of rules, that satisfy the property of dominance, for ranking sets in terms of diversity. However, all these contributions take into account a prior notion of proximity or dissimilarity.[3]

In the last twenty years, interest in diversity measurement has also arisen in a non-welfarist normative economics in connection with the issue of comparing opportunity sets on the basis of freedom of choice that they offer (see Sugden [18] and Barberà, Bossert and Pattanaik [3] for a survey of the literature). However this literature presents a major weakness as it is insensitive to the diversity of the options contained in opportunity sets.

But what does diversity mean for us? It should be clear that the word diversity is used in different context and with different interpretations. For us, diversity focuses on the differences of endowments among individuals. Our approach finds a natural application in poverty or social exclusion measurement. In additional, the measurement of diversity, using an overall index consent to us to evaluate if there are some change in the degree of diversity over the time, following up a political implementation oriented to reducing diversity.[4]

The present paper is primarily concerned with the axiomatic characterization of a diversity index and a diversity ordering based on the count of attributes in which individuals differ. We show that there exists an index that represents this ordering. In fact, as emphasized in utility theory, a (diversity) ordering may be represented by distinct indices and there exit also (diversity) orderings that are not captured by any index.

We assume that, for each society, it is possible to define a so called *diversity matrix* that is a matrix whose elements indicate the diversity between two individuals of the society. This matrix represents the primary input of our analysis.

The construction of this matrix is quite simple. We consider a matrix that represents the distribution of some characteristics among individuals and we compare each individual with all the individuals in the society and we store in the diversity matrix the degree of diversity among all the pair of individuals. Therefore, the main idea is to use a distance that counts the number of attributes in which all the individuals differ. So, we define in a diversity context the well known counting approach formulated by Atkinson [2] in a context of deprivation. In that work, he suggests an alternative way to define deprivation simply counting the number of dimensions in which each persons is deprived. The counting approach is a suitable method even if variables are ordinal and categorical. Chakravarty and D'Ambrosio [8] and Bossert, D'Ambrosio and Peragine [6] modify this methodology applying in a contest of social exclusion and deprivation and taking into account different weights for different dimensions. Alkire and Foster [1] apply the counting approach in a multidimensional poverty context

[3]See Nehring and Puppe [13] for a critical review.

[4]In this case we suppose that diversity has a negative meaning.

as an intermediate approach between the intersection and the union approach[5] in defining poverty. In their work, they propose an average of the number of deprivation suffered by the poor. Also Lasso de la Vega [11] propose deprivation curves in order to establish dominance criteria to measure multidimensional poverty in a counting approach.

Given a diversity matrix it is possible to define a Diversity Measure that is a function that associates to each diversity matrix a real number that represents the degree of diversity of the society. This function fulfills the Monotonicity Axiom, the Normalization Axiom and the Anonymity Axiom. So we construct a diversity index for a society that helps us to make comparison among different societies.

Then we introduce a two stage aggregation procedure: we first focus on the construction of a individual diversity index that counts, for a fixed number of characteristics, the degree of diversity of the $i-th$ individual respect the society in which he/she lives. Using the definition of diversity score it is possible to define a diversity profile for a given society. Then, we move from a vector representation of the society to a summary statistic following an aggregation procedure.

The two-stage procedure described above, is the usually way used to construct aggregate measures in a multidimensional framework. Dutta, Pattanaik and Xu [10] provide a detailed description of these two stages. Bossert, D'Ambrosio and Peragine [6] and Bossert, Chakravarty and D'Ambrosio [5] propose a two different step in aggregation in a context of deprivation and social exclusion and multidimensional poverty and multidimensional material deprivation respectively.

Using the definition of overall diversity index we introduce a Counting Diversity Ordering as the ordering represented by this index. In order to characterize our ordering we introduce two axioms. The first one is a Monotonicity Axiom. It requires that if we have two society with only two individuals, then the ranking is simply determined by computing the diversity between the two individuals. The second axiom we propose is a Separability Axiom. It requires that under some circumstances the ranking of two distinct societies does not change if we add in each society a group of identical individuals such that they have the same degree of diversity in the new society in which the group is added.

The rest of the paper is organized as follows. The next section is devoted to the introduction of the basic notations and the formal definitions. Section 3 presents the characterization of the two steps of aggregation of the index. We also introduce some additional properties that our index satisfies and the relationship with the relate literature In section 4 we suggest some axioms that characterize a diversity ordering. and we prove our characterization results. Section 5 concludes our works.

## 2. Definitions and Notations

Let $\mathbf{X} = [x_{ij}]$ be the $n \times k$ matrix that represents a multidimensional distribution of $k$ attributes among $n$ ($n \geq 2$ and $k \geq 1$) individuals in a society and let us assume that $x_{ij} \in \mathbb{R}_+$. With this notation, the $j-$th column of the $X$ matrix represents the distribution of the $j-$th attribute among the $n$ people of the society, while the $i-$th row gives the vector of $k$ attributes for individual $i$.

Attributes might have different interpretations according to the specific context. Indeed they can represent health, income, life expectancy, scholarship, right to vote, sex, religion or education and so on.

[5]Following the intersection approach, an individual is identified as poor if he fall in poverty, or equivalently he is deprived, in all dimensions. Whereas, with the union approach, people is defines as poor if he is deprived in at least one dimension. Obviously this second approach is very restrictive, in fact when the number of attribute is large, the union approach will often identify most of the population as being poor. The notion of union and intersection approach was first formulated by Atkinson [2].

Moreover, attributes can be represented by quantitative, ordinal or even categorical variables. This is a crucial point in our results. In fact, sometimes, if we change the variables in which attributes are measured, for instance in a context of poverty measurement, the results change or, there exist some metrics that can be used only if data are quantitative. In our context we are able to manage integer value, for example if we take in account years of education or life expectancy, but data could also be categorical for instance if we refer to the level of education (that it could be classified in high level, middle level or low level) or we could work with 0/1 value, for instance if the attributes are right to vote or sex. We also could use all those distinct kinds of data at the same time without compromise our analysis.

Let also assume that these attributes can be owned simultaneously by multiple individuals (that is, they are non-rival).

In additional we suppose that the elements of our matrices are 0 or 1. If the attributes are real numbers, we assume to be able to turn in 0/1 using an exogenous cut-off $\mathbf{z} = (z_1, z_2, ..., z_k)$ such that, for all $i \in \{1, ..., n\}$

$$x_{ij} = \begin{cases} 0 & \text{if } x_{ij} \geq z_j \\ 1 & \text{if } x_{ij} < z_j \end{cases}$$

for all $j \in \{1, ..., k\}$

We indicate with $M(n, k)$ the class of all $n \times k$ matrices and let us to denote with $\mathbf{M}$ the set of all the finite matrix over $M(n, k)$

$$\mathbf{M} = \bigcup_{n \geq 2} \bigcup_{k \geq 1} M(n, k)$$

For all $\mathbf{X} \in M(n, k)$ it is possible to obtain a so called *diversity matrix* $\mathbf{D}_X \in \mathbb{Z}_+^{n,n}$ with the procedure explained below.

The construction of this matrix is quite simple and it derives from the comparisons among individuals. In fact, we compare each individual, represented by a $k-$dimensional vector, with all the individuals (vectors) in the society (matrix).

In order to evaluate this degree of closeness among individuals, we use a function $d$ such that for any individual $i$ and $j$ endowed with the same number of attributes

$$d(i, j) = d_{ij}$$

represents the level of diversity between $i$ and $j$.

We require that this functions satisfies some properties. The first property we require affirms that if we have two identical individual bundles $i$ and $j$ where a generic bundle for individual $l$ is a row vector of the $\mathbf{X}$ matrix $\mathbf{x}_{l\cdot} = (x_{l1}, x_{l2}, ..., x_{ln})$, then this function assumes a null value

$$\forall i, j \in \mathbf{X},\ i = j \text{ then } d_{ij} = 0$$

We require that the function should assume positive value if individuals are distinct

$$\forall i, j \in \mathbf{X},\ i \neq j \text{ then } d_{ij} > 0$$

It also seems reasonable to require that there is a symmetry between pairs of individuals

$$\forall i, j \in \mathbf{X},\ d_{ij} = d_{ji}$$

Then a function that satisfies these properties is a distance function or dissimilarity function. In additional we assume that the triangular inequality holds

$$\forall i, j, l \in \mathbf{X},\ d_{ij} + d_{jl} \geq d_{il}$$

then the function $d$ is a metric.

Anyway, not all the distances allows us to pursue our aim. We use a very particular distance function such that, let $\mathbf{x}_i$ and $\mathbf{x}_j$ denote respectively two $k-$dimensional vectors that represent the endowments of individuals, the diversity between $i$ and $j$ is:

$$d_{ij} = \# \{h | x_{ih} \neq x_{jh}\} \tag{2.1}$$

As formulated $d_{ij}$ only takes in account the number of attributes in which individuals differ.

It is easy to check that this kind of distance satisfies the properties of a distance.[6]

So doing, each entry $d_{ij}$ of the $\mathbf{D}_X$ matrix represents the number of attributes in which individuals $i$ and $j$ differ.

These matrices obtained with this procedure are symmetric matrices with the diagonal equal to zero. We will denote by $\mathbf{d}_i \in \mathbb{Z}_+^n$ the row which corresponds to the diversity of individual $i$ respect to the rest of the society. It is also easy to check that each elements of the $\mathbf{D}_X$ matrix is in the range $0 \leq d_{ij} \leq k$.

We assume that, for each society we can build this diversity matrix that is the primary input of our analysis. We observe that with this kind of aggregation we loose the multidimensional framework. In fact, using the our distance, we implicitly assume that a first aggregation step has already been performed in order to compute $d_{ij}$.

This not surprising, in fact we are only focused on the number of attributes (in our case, the number of attributes in which two individuals differ) and not in which characteristic they differ. As a matter of fact, we suppose that all the attributes are primary good or essential functioning. In additional, we observe that if we are interesting in aggregating using any weight (each dimensions has different importance) then it is possible to generalize the definition of the diversity matrix and define $\mathbf{D}_X \in \mathbb{R}_+^{n,n}$.

However, not all the symmetric and with null elements on the principal diagonal matrix are allowed. First of all, as mentioned above, the values that each element of a diversity matrix assume are in the range $[0, k]$ where $k$ is the number of characteristics. Second, since each element of the matrix represents the number of attributes in which an individual differs from another, if we set some values, then the others must necessarily assume specific values.

So doing let us define $D$ the set of all these feasible matrices.

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[6] From a mathematical point of view, in order to make comparison among these vectors, we used the so called *Hamming distance.* Let $\mathbf{x}, \mathbf{y} \in \{0,1\}^k$ be respectively the vectors of two distinct individuals with $i \in \{1, ..., k\}$ attributes, we have

$$H(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{k} |x_i - y_i| = |\{i : x_i \neq y_i\}| \quad \forall i \in \{1, ..., k\}$$

The name "Hamming distance" is due its inventort Richard Hamming who introduced it in his fundamental paper on Hamming codes "*Error detecting and error correcting codes*" in 1950. This distance is used in telecommunication to count the number of flipped bits (that is the bits that have changed) in a fixed-length binary word. From an information theory point of view, the Hamming distance is equal to the number of ones in $\mathbf{x}$ XOR $\mathbf{y}$ and the metric space of length-$k$ binary strings, with the Hamming distance, is known as the Hamming cube; it is equivalent as a metric space to the set of distances between vertices in a hypercube graph.

**Remark 1.** *Using our Distance, we implicitly are assuming that if we define a new matrix* $\mathbf{Y}$ *obtained from* $\mathbf{X}$ *by a permutation of attributes, then the diversity matrix remains unchanged. More formally, if* $\mathbf{X} \in M(n,k)$ *and* $\mathbf{P}$ *is a* $(k \times k)$ *permutation matrix, the* $\mathbf{D}_X = \mathbf{D}_Y$ *where* $\mathbf{Y} = \mathbf{XP}$*. This assumption may appear strong. Nevertheless, as stressed in the previous pages we are using a specific distance that only count the number of attributes that are diverse. Then, for us, it is irrelevant if two individuals are diverse in health and income or in income and education because in both cases we obtain the same degree of diversity. This is the same in the Social Exclusion or Deprivation Measurement, if fact if we define, as in a Deprivation Context, attributes in a dichotomic manner we come to the same results.*

In additional we suppose that the neither the number of individuals nor the number of attribute is fixed. This means that it is always possible to make comparisons among societies with different value. Nevertheless we observe that, in order to establish unanimous ranking, not all the attributes are relevant. In other words it is possible to add some attributes that are equal for all the individuals without reverse the ranking. The following example would clarify our idea.

**Example 1.** *Let us suppose to establish a ranking of two societies that represent two European Countries, for instance I=Italy and S=Spain. We choose three*[7] *relevant attributes: health, years of educations and income level. Then we can be able to affirm in which countries there is a greater level of diversity. If we introduce the nationality and we compute the degree of diversity, in both the case we obtain the same value*

In this first example, it makes no sense introduce as relevant attribute the nationality of the individuals because it's hardly surprising that all the individuals in society I and S have the same nationality.

Now we consider this second situation

**Example 2.** *Let us suppose to rank groups of individuals in the European Union, for instance we would to compare man and woman in order to establish in which group there is more diversity. Let us suppose to take in account firstly only the three attributes introduced in the previous example and then to bring in also nationality. In this case we could have different ranking.*

In this second example the attribute Nationality plays the same role of all the rest of attributes we choose in the first example, so Nationality is a relevant attribute that helps to define in which group there is a greater or lower level of diversity.

Now we would formalize these situations. Let us consider the first one in a simple way, assuming that we have only two individuals with three attributes. As stressed above, we introduce an attribute that is not relevant or equivalently an attribute that it is equal for both the individuals. Then, the degree of diversity is the same in both situations.

This means that somehow we can always go from the first situation (a society with two individuals with three relevant attributes) to the second one (two individuals with four attributes) adding in the matrix of attributes a column with each components takes the same value (and consequently add a null row and column in the diversity matrix). We observe that we also could think in terms of removing the common attributes as follows.

Let $\mathbf{X}' \in M(2,k)$ and suppose that there exists an attributes that takes the same value for each individual. Then

$$\mathbf{D}_{X'} = \mathbf{D}_X$$

[7]For the following example the number of attributes does not play role.

where $\mathbf{D}_{X'}$ indicates the diversity matrix associated to society $\mathbf{X}'$ in which individuals are compared respect to the $k$ attributes and $\mathbf{D}_X$ represents the diversity matrix associated to society $\mathbf{X}$ in which individuals are compared respect to the $k-1$ attributes, removing the identical attribute.

We introduce some additional formalizations. We firstly define a diversity measure as follows.

**Definition 1** (Diversity Measure S)**.** *A diversity Measure for a given society is a function defined on the space of all the diversity matrices $D$*

$$S{:}D \to \mathbb{R} \tag{2.2}$$

*that associates with every diversity matrix $\mathbf{D}_X$ obtained using the distance defined in equation (2.1) a real number $S_X$*

$$S(\mathbf{D}_X) = S_X$$

*that represents the degree of diversity of the society.*

According to this definition we are implicitly assuming that the overall diversity measure in a society $\mathbf{X}$ depends only on the number of characteristics in which the individuals differ.

Then, the diversity measure we propose is a real valued function of the degree of diversity among individuals in a given society. This measure fulfills some axioms: Monotonicity (MONO), Anonymity (ANON) and Normalization (NORM).

**MONO:** Monotonicity Axiom, in a diversity context, simply requires that an overall index of diversity increases if the value of diversity for some individual increases with no decrease for any individual. Using our notation, we have

$$S_X \geq S_Y \Leftrightarrow d_{ij}^X \geq d_{ij}^Y$$

This axioms is a standard axiom in economic literature. For instance, if we apply this axiom in a welfare framework we are assuming that each attribute of well-being contributes positively to social welfare. In an unidimensional poverty measurement, the axiom means that the reduction in the income of a poor individuals must increase the poverty index. Finally, in Social Exclusion Measurement this axiom requires the measure to increase if the deprivation score of an individual increases.

**ANON:** Anonymity Axiom asserts that in order to compute our index, all the individuals are equivalent. In other words, if we change the position of the individuals then the overall index remains the same

$$S_X = S_Y \Leftrightarrow \mathbf{Y} = \pi(\mathbf{X})$$

where $\pi$ is function that permutes the rows of the $\mathbf{X}$ matrix.

**NORM:** The Normalization axiom ensures that if among individuals there is no diversity the measure reflect this absence of diversity. In other words, if all the elements of the diversity matrix are 0, then the overall index is 0.

$$S_X = 0 \Leftrightarrow \mathbf{D}_X = \mathbf{0}$$

where $\mathbf{0}$ indicates the null matrix.

For all $i \in \{1, ..., n\}$ we denote with $\mathbf{d}_i$ the correspondent $n$ dimensional row vector of diversity matrix $\mathbf{D}_X$

$$\mathbf{d}_i = (d_{i1}, d_{i2}, ..., d_{in})$$

where each element $\mathbf{d}_{ij}$, $j \in \{1, ..., n\}$ represents the diversity score between individual $i$ and individual $j$. The standard way to move from this $i-$th $n$ dimensional vector to a single number (index) that represents the degree of diversity of the overall society is by using two different aggregation procedure. Dutta, Pattanaik and Xu [10] refer to this way as Procedure I. This procedure consists in two different step: the first one focus on calculating a single individual measure and the second one on obtaining the overall diversity index.

In order to obtain the single individual measure, we define, for any individual, a function $f$[8] as follows:

$$f : \{0, ..., k\}^n \to \mathbb{R}_+ \tag{2.3}$$

such that $s_i^n = f(\mathbf{d}_i)$. This function aggregates $(d_{i1}, d_{i2}, ..., d_{in})$ into $s_i^n$ for all $i \in \{1, ..., n\}$.

So doing, for all diversity matrix $\mathbf{D}_X$ we obtain a vector of individual diversity

$$\mathbf{s}_X^n = (s_1^n, s_2^n, ..., s_n^n)$$

where subscript $\mathbf{X}$ reminds us that this vector is associated to the $\mathbf{X}$ society and hence to the $\mathbf{D}_X$ matrix, while the superscript $n$ reminds us the number of individuals in the society.[9] Using the definition of the previous section, this vector is the diversity profile for a society $\mathbf{X}$.

The second aggregation step requires to define a function $g$

$$g : \mathbb{R}_+^n \to \mathbb{R}_+ \tag{2.4}$$

to aggregate

$$s_1^n = f(d_{11}, d_{12}, ..., d_{1n}), ..., s_n^n = f(d_{n1}, d_{n2}, ..., d_{nn})$$

into an overall index of diversity. This function aggregates $\mathbf{s}_X^n = (s_1^n, s_2^n, ..., s_n^n)$ into $S_X^n$, in other words, we move from a $n$ dimensional vector to a single value that is our diversity index.

Thus, our two stage aggregation procedure uses two functions $f$ and $g$ the first one that gives us the overall individual diversity for individual $i$ once we know his diversity vector $\mathbf{d}_i$ and the second one that gives us the overall diversity of the society once we know the overall deprivation of each individual:

$$S_X^n = g\left(f(d_{11}, d_{12}, ..., d_{1n}), ..., f(d_{n1}, d_{n2}, ..., d_{nn})\right)$$

## 3. The index

We propose a characterization of the two steps of aggregation.

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[8]This function is the same among all the individuals, in other words

$$f \equiv f_i \quad \forall i \in \{1, ..., n\}, \; f_i : \{0, ..., k\}^n \to \mathbb{R}_+$$

in fact in order to built the individual diversity measure $s_i^n$ we make a weighted sum of the components of the diversity vector $\mathbf{d}_i$ and this procedure is the same for all individuals.

[9]This notation, though not particularly easy to use, it will be necessary over the next paragraphs in which we present a characterization of $s_i^n$ and $S_X^n$. If this does not confuse, we will use $s_i$ insteed of $s_i^n$.

3.1. **The individual measure: $s_i^n$.** We introduce a standard aggregation (first stage) following the counting approach (see, among others, Atkinson [2], Chakravarty and D'Ambrosio [8], Bossert, D'Ambrosio and Peragine [6]). So doing we can define an individual diversity vector as an element of $d_{ij} \in \{0, 1, ..., k\}^n$ (i.e. a row of the matrix).

**Definition 2.** *An individual measure of diversity $s_i^n$ for the individual $i$ is the aggregate degree of diversity of itself respect to the $n-$society in which he/she lives.*

In other words, an individual diversity measure $s_i^n$ for individual $i$ respect to the society is, for all $i \in \{1, ..., n\}$ a function that maps each vector of diversity in a positive real number. So doing we move from a matricial framework to a vector one. The elements of this vector represent the degree of diversity of the $i - th$ individual, for all $i \in \{1, ..., n\}$.

Then, for all individual, $s_i^n$ is the weighted sum of the degree of diversity among one individual and the rest of the society.

In order to characterize this measure, we introduce some axioms.

The first one is the so called Anonymity or Symmetry Axiom. It asserts that for individual $i$, the individual diversity associated with $\mathbf{D}_X$ and $\mathbf{D}_Y$ is the same if $\mathbf{D}_Y$ is obtained from $\mathbf{D}_Y$ by a permutation of individuals that leave unchanged their attributes. In other words, if two or more persons switch achievements, the individual diversity is the same.

**Axiom 1** (Anonymity (A))**.** *Let $\mathbf{D}_X \in \mathbb{R}^{n\times n}$ be the diversity matrix associated to a real matrix $\mathbf{X}$ $\in \mathbb{R}^{n\times k}$ and $\mathbf{D}_Y$ the diversity matrix associated to a new real matrix $\mathbf{Y} = \mathbf{PX}$ where $\mathbf{P}$ is a $n \times n$ permutation matrix, then*

$$s_i^{\mathbf{X}} = s_{\pi(i)}^{\mathbf{Y}}$$

*and $\pi(\cdot)$ is a permutation function over individuals.*

In additional this axiom requires that does not matter if the individual $i$ differs from individual $j$ or individual $l$ in some characteristics because we aggregate over all the individuals. In fact

$$s_{\pi(i)} = f\left(d_{\pi(i),1}, d_{\pi(i),2}, ..., d_{\pi(i),n}\right)$$

**Condition 1.** *The function $f \equiv f_i \quad \forall i \in \{1, ..., n\}$ is invariant under permutation.*

Second axiom suggests how to obtain our index if in the society there are only two individuals. Let us suppose that a given society consists only in two individuals $n = \{1, 2\}$. The diversity matrix $\mathbf{D}_X$ $\in \mathbb{R}_+^{2;2}$ takes the following form

$$\mathbf{D}_X = \begin{pmatrix} 0 & d_{12} \\ d_{21} & 0 \end{pmatrix}$$

By definition of $d_{ij}$ we can affirm that $d_{12} = d_{21}$ (in fact $\mathbf{D}_X$ is a square symmetric matrix). It follows that the two vector are identical

$$\mathbf{d}_1 \equiv \mathbf{d}_2$$

and they have also a null component (that represents the diversity between an individual and its self. Then it seems reasonable to require that the individual diversity index the not null value.

**Axiom 2** (Two Individual (2I))**.** *For $n = 2$, $n = \{1, 2\}$*

$$s_1^2 = s_2^2 = d_{12} = d_{21}$$

Next axiom suggests how calculate the index if we add a new individual.

**Axiom 3** (Add one individual (AI))**.** *For all* $n \geq 3$ *and* $i = 1, ..., n-1$

$$s_i^n = s_i^{n-1} + d_{in}$$

This axiom affirms that, if we have a society with at least three individuals (let us suppose that we have $n-1$ individuals and then the correspondent individual diversity measure is $s_i^{n-1}$) and we would add a new individual then the new individual diversity measure individual $i$ in this new society with $n$ individuals is the sum of the individual diversity measure with $n-1$ people ($s_i^{n-1}$) and the diversity between this individual and all the members of the new society.

As defined in the statement of the axiom it is valid $\forall i = 1, ..., n-1$ but if we require that the index also must fulfills Anonymity Axiom we can obtain a way to built the individual diversity measure for individual $n$.

These properties allow us to characterize our individual measure of diversity $s_i^n$:

**Proposition 1.** *An individual diversity measure* $s_i^n$ *satisfies Anonymity Axiom (A), Two Individual Axiom (2I) and Add one individual Axiom (AI) if and only if it takes the following form:*

$$s_i^n = \sum_{j=1}^{n} d_{ij} \tag{3.1}$$

*Proof.* That the measure as defined above satisfies Anonymity Axiom (A), Two Individual Axiom (2I) and Add one individual Axiom (AI) is straightforward to check.

Let us suppose now that $s_i^n$ satisfies (A)- Axiom (2I)-Axiom and (AI)-Axiom. We must prove that $s_i^n$ is the sum of the degree of diversity among $i$ and the rest of individuals in the society.

We use induction on the number of individuals ($n$).

If $n = 2$ (base case), let us suppose $n = \{j, k\}$, $j \neq k$ and $\mathbf{d}_j, \mathbf{d}_k \in \mathbb{R}_+^2$ . By symmetry of $\mathbf{D}_X$ matrix we have $\mathbf{d}_j \equiv \mathbf{d}_k$. By (2I)-Axiom we have, for all $i \in \{j, k\}$ :

$$s_i^2 = d_{jk} = \sum_{j=1}^{2} d_{ij}$$

We assume that (3.1) holds $\forall n$ (inductive step). We prove that it holds for all $n+1$.

By (AI)-Axiom and (A) Axiom (that they hold $\forall n$) we can write:

$$s_i^{n+1} = s_i^n + d_{i,n+1}$$

by induction hypothesis we can replace $s_i^n$

$$= \sum_{j=1}^{n} d_{ij} + d_{i,n+1} = \sum_{j=1}^{n+1} d_{ij}$$

which completes the proof. $\square$

3.2. **Some additional properties that $s_i^n$ satisfies.** It is easy to check that our individual diversity measure fulfills some additional properties.

First of all, $s_i^n$ has a lower bundle. In fact it takes value in $[0, (n-1)k]$. The next property is devote to define this range:

**Property 1** (Normalization)**.** $\forall i, j \in \{1, ..., n\}$*, if* $d_{ij} = 0$ *then* $s_i^n = 0$.

This property asserts that if, for the $i-th$ individual there is no diversity respect to the other individuals in the society, then the value of the index (i.e. the aggregate degree of diversity) must be zeros.

Otherwise, if there is a complete diversity between the $i-th$ individual of the society ($d_{ij}=k$ $\forall j\in\{1,...,n\}\setminus\{i\}$, where $k$ is the number of attributes and $n$ the number of individuals) then $s_i^n$ must assume the maximum value: $s_i^n=(n-1)k$.

Furthermore, our individual measure of diversity must be invariant under proportional change. The following property tries to define this peculiarity. It asserts that if $\mathbf{Y}$ is obtained from $\mathbf{X}$ by a proportional change (i.e. if every attribute of $\mathbf{Y}$ is obtained multiplying each attribute of $\mathbf{X}$ by a scalar $\alpha>0$), the respective diversity matrix $\mathbf{D}_Y$ does not change ($\mathbf{D}_X=\mathbf{D}_Y$) and then the individual diversity measure associated with this new distribution is the same associated with $\mathbf{X}$.

**Property 2** (Scale invariance (Zero-Degree Homogeneity))**.** *For all $\alpha\in\mathbb{R}_{++}$, if $\mathbf{Y}=\alpha\mathbf{X}$, then $\mathbf{D}_X=\mathbf{D}_Y$ and $s_i^{\mathbf{X}}=s_i^{\mathbf{Y}}$.*

**Property 3** (Diversity)**.** *If $\mathbf{D}_Y$ is obtained from $\mathbf{D}_X$ by a change in the $\mathbf{X}$ matrix of a given characteristic $k$, for a fixed individual $i$ such that $\forall l\in\{1,...,n\}$*

$$x_{ik}\neq x_{lk}\quad \textit{and}\quad y_{ik}\neq y_{ik}$$

*then*

$$s_i^X=s_i^Y$$

This property requires that if a change occurs among individual that for a given characteristic are diverse and after this change they remain diverse too, then the individual measure of diversity does not change, and also the diversity matrix remains the same ($\mathbf{D}_X=\mathbf{D}_Y$)

**Property 4** (Population Proportionalu Invariance)**.** *If $\mathbf{D}_Y\in\mathbb{R}^{tn,tn}$ is obtained from $\mathbf{D}_X\in\mathbb{R}^{n,n}$ by a $t-$time replication of all the individuals of the original matrix $\mathbf{X}$, then*

$$s_{i+T}^Y=ts_i^X$$

*where $T=0,1,...,t$.*

This property asserts that if $Y$ is a $t-$replication matrix of $\mathbf{X}$ in the sense that $\mathbf{Y}$ is obtained from $\mathbf{X}$ replicating $t$ times the $\mathbf{X}$ matrix, and the respective diversity matrices are $\mathbf{D}_Y\in\mathbb{R}^{tn,tn}$ and $\mathbf{D}_X\in\mathbb{R}^{n,n}$ where $\mathbf{D}_Y$ is a blocks matrix of the form

$$\mathbf{D}_Y=\underbrace{\begin{pmatrix}\boxed{\mathbf{D}_X} & ... & \boxed{\mathbf{D}_X}\\ ... & & ...\\ \boxed{\mathbf{D}_X} & ... & \boxed{\mathbf{D}_X}\end{pmatrix}}_{t-\text{times}}$$

then it is possible to obtain the new individual diversity index by multiplying the old individual value for the number of times in which we would replicate. This axiom is the counterpart of the Replication Invariance Axiom in poverty measurement or in social exclusion or in deprivation context. In particular, in the latter case, Bossert, D'Ambrosio and Peragine [6] also obtain a factor scale invariance that depends on the square of $t$. It is also know in unidimensional poverty measurement as the Population Principle Axiom formulated in 1920 by Dalton [9]. In this work, Dalton requires that an income distribution is to be regarded as distributional equivalent to a distribution composed by replication of it.

**Property 5** (Monotonicity)**.** *Let* $\mathbf{D}_X$ *and* $\mathbf{D}_Y$ *the diversity matrix associated respectively to* $\mathbf{X}$ *and* $\mathbf{Y}$*, if for a fixed individual* $i$*, and* $\forall j \in \{1, ..., n\}$*,*

$$d_{ij}^X \geq d_{ij}^Y \quad \textit{then} \quad s_i^X \geq s_i^Y$$

This property asserts that if for an individual $i \in \{1, ..., n\}$ each components of the individual diversity vector $\mathbf{d}_i^X$ is greater than the correspondent individual vector $\mathbf{d}_i^Y$ then the individual diversity measure associated to $\mathbf{D}_X$ must be greater than $\mathbf{D}_Y$. In other words, let us consider a vector of the $\mathbf{D}_X$ matrix

$$\mathbf{d}_i = (d_{i1}, d_{i2}, ..., d_{ij}, ..., d_{in}, )$$

and let $\delta \in \mathbb{R}_{++}$ a real positive non null value. Let us suppose to obtain from $\mathbf{d}_i$ a new diversity vector $\mathbf{d}_i'$

$$\mathbf{d}_i' = (d_{i1}', d_{i2}', ..., d_{ij}' + \delta, ..., d_{in}')$$

then the Monotonicity Property requires that the individual diversity in $\mathbf{d}_i'$ is higher than $\mathbf{d}_i$.

3.3. **Aggregate measure:** $S_X^n$**.** Now, we move from an individual diversity measure $s_i^n$ for each individual in a society to an aggregate diversity measure $S_X^n$. This is our second step aggregation.

As introduced in the previous pages $S_X^n$ associates to all the $n$ dimensional vectors obtained following the first aggregation step a positive real value:

$$S_X^n : \bigcup_{n \in \mathbb{N}} \mathbb{R}^n \to \mathbb{R}_+$$

In other words, this overall index gives the degree of diversity of the society **X.**

In order to characterize this measure and following the first stage of aggregation, we introduce some axioms.

The first axiom we require is the so called Anonymity Axiom. It asserts that the overall diversity index does not change if we permute the individual diversities.

**Axiom 4** (Anonymity* (A*))**.** $\forall n \in \mathbb{N}$*,* $n \geq 2$*, let* $\mathbf{s}^X$ *the individual diversity vector and* $\mathbf{s}^Y = \sigma(\mathbf{s}^X)$ *where* $\sigma$ *is a permutation of elements of the vector, then*

$$S_X^n = S_Y^n$$

In other words we obtain the same value of diversity if we permute the elements of the vector of diversity $\mathbf{s}^X$ .

The second axiom we introduce shows how to calculate the overall index in the society there are only two individuals. Obviously, if we have only two individuals, by the (2I) Axiom, we have:

$$s_1^2 = s_2^2 = d_{12} = d_{21}$$

than is a reasonable request that the overall diversity index is a average value of the individual diversity measure for the two individuals.

**Axiom 5** (Two individual* (2I*))**.** *If* $n = 2$*,* $n = \{1, 2\}$

$$S_{\mathbf{X}}^2 = (s_1^2 + s_2^2) = 2s_1^2$$

Using (2I) Axiom, the (2I*) Axiom could be wrote as

$$S^2_{\mathbf{X}} = s^2_1 + s^2_2) = d_{12} + d_{21} = 2d_{12}$$

This axiom requires that, if in a given society $\mathbf{X}$ there are only two individuals, the the degree of diversities given by the sum of the individual measure of diversity.

Obviously if $d_j = d_k = 0\ \forall j, k \in\{1, ..., n\}$ (i.e. the individual have the same null vector of diversity or, in other words, the individual are identical) then $S^n_{\mathbf{X}} = 0$.[10]

**Axiom 6** (Add an individual on* (AI*))**.** $\forall n \in \mathbb{N}$, $n \geq 3$

$$S^n_{\mathbf{X}} = S^{n-1}_{\mathbf{X}} + 2s^n_n$$

*where $s^n_n$ is the individual measure of diversity oh the $n-th$ individual when in society there are $n$ individuals.*

This axiom asserts that, if we add an individual, the overall measure of diversity is given by the sum of diversity before add this individual and the degree of diversity of oneself.

**Proposition 2.** *An aggregate measure of diversity $S^n_{\mathbf{X}}$ satisfies Anonymity* (A*), Two Individual* Axiom (2I*) and Add an individual on* (AI*) Axiom with individual diversity measure $s^n_i$ that fulfills Anonymity (A), Two Individual Axiom (2I) and Add an individual on (AI)if and only if for all $s^n_i \in \mathbb{R}^n$*

$$S^n_{\mathbf{X}} = \sum_{i=1}^{n} s^n_i \tag{3.2}$$

*Proof.* It is straightforward to check that the measure $S^n_{\mathbf{X}}$, as defined, satisfies the set of required axioms.

We consider the other implication: take any diversity measure $S^n_{\mathbf{X}}$, let us suppose that it satisfies Anonymity*Axiom (A*), Two Individual* Axiom (2I*) and Add an individual on* (AI*) Axiom. We must prove that $S^n_{\mathbf{X}}$ is a normalized sum of the individual measures of diversity.

We use inductions the cardinality of $n \in \mathbb{N}$.

If $n = 2$ (base case), suppose $n = \{i, j\}$, $i \neq j$ and $d_i, d_{j \in} \mathbb{R}^2$. By (2I*) Axiom we can write

$$S^n_{\mathbf{X}} = s^2_j(d_j) + s^2_i(d_i) = \sum_{i=1}^{2} s^2_i$$

We assume that (3.2) holds for all $n$ (inductive step) and we show that it holds for $n + 1$.

By (AI*) Axiom, that holds for all $n$, we have

$$S^{n+1} = S^n_{\mathbf{X}} + 2s^{n+1}_{n+1}$$

and by induction hypothesis we have

$$S^n_{\mathbf{X}} = \sum_{i=1}^{n} s^n_i = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} d_{ij}\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}$$

and we can replace in the $S^{n+1}$ expression:

$$S^{n+1} = S^n_{\mathbf{X}} + 2s^{n+1}_{n+1} = \sum_{i=1}^{n} s^n_i + 2s^{n+1}_{n+1}$$

[10]This derive for the normalization property that $s_i$ fulfills.

By proposition 1, the last equation can be rewritten as

$$S^{n+1} = \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij} + 2\sum_{j=1}^{n+1} d_{ij}$$

and rearranging we complete the proof. □

3.4. **Some additional properties that $S^n_{\mathbf{X}}$ satisfies.** This subsection presents some additional properties that this aggregate index satisfies. It is quite easy to prove that the aggregate diversity index satisfies some[11] of the properties introduced individual index. In particular, also the aggregate measure of diversity fulfills the Normalization property:

**Property 6** (Normalization). *$\forall i \in \{1, ..., n\}$, if $s^n_i = 0$ then $S^n_{\mathbf{X}} = 0$.*

As in the individual diversity measure, this property asserts that if in the society there is no diversity, namely, for each individual, his individual diversity measure is zero, then it is obvious that also the aggregate value should take zero. Otherwise, if there is a complete diversity ($s^n_i = (n-1)k$ $\forall i \in \{1, ..., n\}$), then $S^n_{\mathbf{X}} = n(n-1)k$.

In addition, we observe that our overall diversity measure (3.2) is similar to the Gini[12] coefficient of inequality

$$G = \frac{1}{2n^2\overline{x}} \sum_{i=1}^{n}\sum_{j=1}^{n} |x_i - x_j| \tag{3.3}$$

where $x$ is the value, $n$ is the number of value observed and $\overline{x}$ is the mean value. In fact also the Gini index it is built taking in account the difference of income distribution by an individual and the rest of individuals in the society. So doing, if in the Gini coefficient we remove the normalization on the square number of individuals and in the mean income we obtain the same range for both the index

It is easy to show that our index of diversity could be interpreted as a special case of generalized Bourguignon and Chakravarty multidimensional poverty family

$$P^{\theta}_{\alpha}(X, \mathbf{z}) = \frac{1}{n}\sum_{i=1}^{n}\left[\left(\sum_{i=1}^{k}\left(w_j a^{\theta}_{ij}\right)\right)^{\frac{1}{\theta}}\right]^{\alpha} \tag{3.4}$$

for a given a multidimensional distribution $X$, a vector of thresholds $\mathbf{z}_k \in \mathbb{R}^k_{++}$, and a vector of weights $\mathbf{w}_k \in \mathbb{R}^k_{+}$, $\sum_{j=1}^{k} w_j = 1$. In additional $\theta > 0$ represents the elasticity of substitution between the normalized gaps of the attributes for any person, $\alpha > 0$ can be interpreted as a sort of society aversion towards poverty or equivalently as a measure of the sensitivity towards poverty and $a_{ij} = \max\left(1 - \frac{x_{ij}}{z_j}, 0\right)$. In fact, if we assume unitary equal weight for all the dimension ($w_j = 1$, for all $j = 1, ..., n$), $\alpha = \theta = 1$. In fact also in this case the Bourguignon and Chakravarty family and our

[11] For instance, the aggregate measure does not satisfy the Scale Invariance Property.

[12] As observed by Sen [16], when $G$ is based on the Lorenz curve of income distribution, it can be interpreted as the expected income gap between two individuals randomly selected from the population. The extreme value of the Gini Coefficient are 0 and 1. The former implies perfect equality (everyone in society has exactly the same amount of income) whereas the latter implies total inequality (one person has all the income and everyone else has nothing). In the first case, all the components of the vector of income are equal to the mean value $(\overline{x}, \overline{x}, ..., \overline{x})$, the $G = 0$. In the second extreme case $(n-1)$ components are zero and the last individual own all the incame $(0, 0, ..., 0, n\overline{x})$ and $G = \frac{1}{2n^2\overline{x}}\left[2n\left(n-1\right)\overline{x}\right] = \frac{n-1}{n} \rightarrow_{n\rightarrow\infty} 1$..

index are obtained following the same procedure. We finally observe that the normalization in poverty measurement is a reasonable request but this is not so obvious in diversity measurement.

We now introduce some additional property of our index. The following property is a interesting property for our index. It addresses the relationship between the overall diversity index for a given society and the overall diversity of various subgroups.

**Property 7** (Decomposability)**.** *Let $\mathbf{D}_X \in \mathbb{R}^{n,n}$ be a diversity matrix, for all $p, q \in \{1, ..., n\}$ such that $p + q = n$ then*

$$S^{p+q} = \sum_{i=1}^{p}\sum_{j=1}^{p} d_{ij} + \sum_{i=p+1}^{p+q}\sum_{j=p+1}^{p+q} d_{ij} + 2\sum_{i=1}^{p}\sum_{j=p+1}^{p+q} d_{ij} \tag{3.5}$$

In inequality measurement, the Decomposability axiom affirms that the index consists of two parts: the first refers to inequality between-group[13] component and the second one, to inequality within-group[14] component. So doing, the index can be express as an exact sum of this two components:

$$I(x) = I_W(x) + I_B(x) \tag{3.6}$$

Moreover in poverty measurement, this axiom requires that the index should be a weighted average of the poverty indices applied to specific subgroup within the population, where the weights are equal to the population share.

In our context, this property requires that the overall diversity measure for a given society $X$ is sum of the diversity among the first $p$ individuals and the last $q$ individuals (with $p + q = n$) and the degree of diversity among individuals in this two distinct group.

If we denote by $\mathbf{X}_p$ and $\mathbf{X}_q$ the submatrix of $\mathbf{X}$ defined by $\mathbf{X}_p = [x_{ij}]_{i,j=1,\dots,p}$ and by $\mathbf{X}_q = [x_{ij}]_{i,j=p+1,\dots,q}$ equation (3.5) may be rewritten as:

$$S^{p+q}_{\mathbf{X}} = \underbrace{S^p_{\mathbf{X}_p} + S^q_{\mathbf{X}_q}}_{\text{within-component}} + 2\sum_{i=1}^{p}\sum_{j=p+1}^{p+q} d_{ij} \tag{3.7}$$

In this way equation (3.7) is very similar to equation (3.6). In fact, the first two terms are the sum of the diversity level of the respective subgroups which may be interpreted as usual as the within-term, assuming that the diversity between the two groups has been removed. The third component is a bit harder to interpret. It would be the diversity level between the groups $p$ and $q$ assuming that the diversity within the respective groups has been removed, that is, there is no more diversity in group $p$ nor in $q$. It would correspond with the between-component as indicated in equation (3.6).

The next property that our diversity index satisfies is the so called Population Proportionality Invariance. This is an important property because, as stressed in the individual diversity measure, it allows us to make comparisons among societies with a different number of individuals.

**Property 8** (Population Proportional Invariance)**.** *Let $\mathbf{s}_n^X$ be the $n-$dimensional vector of individual diversity and*

$$\mathbf{s}^Y_{nt} = \underbrace{(\mathbf{s}^X_n, ..., \mathbf{s}^X_n)}_{t-times}$$

[13]Obtained by imagining that each person in any subgroup receives the subgroup's mean income.

[14]Obtained as a weighted sum of subgroup inequality levels, the weights depending on the subgroups' income shares or population shares or some combination of the two shares.

*then*

$$S_{nt}^{Y} = t^2 S_n^X$$

This property asserts that $\mathbf{s}_{nt}^{Y}$ is a new individual diversity vector obtained by replicating the $\mathbf{s}_n^X$ vector, or similarly by replicating each elements of the $\mathbf{s}_n^X$ vector, $t-$times, then the overall diversity index associated to this new vector it is simply the product of the old diversity index associated with $\mathbf{s}_n^X$ and the square of the time of the replications

The last property we mention is the Monotonicity Property. This property asserts that if an element of the individual diversity vector increases then the overall diversity index increases.

**Property 9** (Monotonicity)**.** *Let $\mathbf{s}_n^X$ and $\mathbf{s}_n^Y$ two individual diversity vectors such that $s_i^X \geq s_i^Y$ for a fixed individual $i \in \{1, ..., n\}$ and $s_j^X = s_j^Y$ for all $j \in \{1, ..., n\}$, $j \neq i$, then*

$$S_n^X \geq S_n^Y$$

This property suggests us that if two vector are identical in each elements but in an elements one is greater than or equal to the correspondent element in the second vector, then the overall diversity index associated to the first individual vector is greater than or equal to the overall diversity index associated to the second individual diversity vector.

## 4. The ordering

In this section we present an axiomatic characterization of a diversity ordering represented by our diversity index. We call the individual diversity measure introduced in Proposition 1 as individual diversity score and we remember that this integer $s_i$ represents the degree of diversity between individual $i$ and the rest of the society in which he lives. Obviously equation (3.1) implies that $s_i$ is an integer value belonging to the following range

$$s_i = \{0, 1, 2..., (n-1)k\}$$

where $k$ represents the number of attributes.

Using the definition of diversity score $s_i = \sum_{j=1}^{n}$ it is possible to define a so called diversity profile in a society $\mathbf{X} \in M(n, k)$ as a $n-$dimensional vector

$$\mathbf{s}_X = (s_1, s_2, ..., s_n)$$

where $s_i$ is the diversity score of the $i-$th individual respect to the rest of individuals in society $\mathbf{X}$.

Finally we introduce the overall diversity measure for society $\mathbf{X} \in M(n, k)$ as the sum of the elements in the diversity profile:

$$S(\mathbf{D}_X) = \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij} \tag{4.1}$$

Combining definition of $s_i$ and equation (4.1)[15] we obtain an analogous expression for the overall diversity measure:

$$S(\mathbf{D}_X) = \sum_{i=1}^{n} s_i \tag{4.2}$$

[15]We observe that equation (4.1) it the same equation proposed by Proposition 2.

According to this definition we are implicitly assuming that the overall diversity index in a society $\mathbf{X}$ depends only on the number of characteristics in which the individuals differ. We observe that $S(\mathbf{s}_X)$ is an even number and it belongs to the range:

$$S(\mathbf{D}_X) \subset \{0, 2, 4, ..., n(n-1)k\} \tag{4.3}$$

Two very simple societies play an important role in our work. The first one is a society with n individuals such that one of them is diverse and the rest of the $(n-1)$ individuals are equal and with only one attribute. Let call $\mathbf{A}$ this society and we denote, as usually with $\mathbf{D}_A$ the diversity matrix associated to $\mathbf{A}$

$$\mathbf{D}_A = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}$$

The diversity profile is

$$s_A = \left( n-1, \ \underbrace{1, ..., 1}_{(n-1) \text{ times}} \right)$$

and the overall diversity index takes the following form: $S(\mathbf{D}_X) = 2(n-1)$. Using this definition we formulate the following Lemma that asserts that it is always possible to find a simple society with the same degree of diversity of a more general one.

**Lemma 1.** *For any $\mathbf{X} \in M(n,k)$ there exists $\mathbf{X}^* \in M(n', 1)$, $\mathbf{x}_i^* = \mathbf{x}^*$ for all $i = 1, ..., n'$ and $\mathbf{x}_i^* = \mathbf{x}^*$ otherwise, such that*

$$S(\mathbf{D}_X) = S(\mathbf{D}_{X^*})$$

*Proof.* The $S_X$ index takes values in the range

$$S(\mathbf{D}_X) \in \{0, 2, 4, ..., n(n-1)k\}$$

Then, there exists an even integer value $d$ in this range such that $S_X = d$.

Now, let us consider a simple society $\mathbf{X}^*$ with exactly $(d/2) + 1$ individuals and one attribute such that differ in $d/2$ of them are equal and one is diverse.

$$\mathbf{X}^* = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Then the diversity matrix associated to this simple matrix is

$$\mathbf{D}_{\mathbf{X}^*} = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}$$

and the value of the overall diversity index is $S(\mathbf{D}_{X^*}) = d$ that completes the proof. $\square$

We observe that $n'$ is always equal to the semi overall diversity index increasing by 1.

$$n' = \frac{S(\mathbf{D}_X)}{2} + 1$$

The second simple society we mention is a society composed by only two individuals that differ in several attributes. Let denote with $\mathbf{B} = \{x, y\}$ this society, the diversity matrix is

$$\mathbf{D}_B = \begin{pmatrix} 0 & d_{xy} \\ d_{xy} & 0 \end{pmatrix}$$

The diversity profile is $\mathbf{s}_B = (d_{xy}, d_{xy})$ and the overall diversity index is $S(\mathbf{D}_B) = 2d_{xy}$. The next Lemma we formulize ensures that there exists a simple society such that it has the same degree of diversity of a more general one.

**Lemma 2.** $\forall$ $\mathbf{X} \in M(n, k)$ *there exists* $\overline{\mathbf{X}} \in M(2, k')$ *such that*

$$S(\mathbf{D}_X) = S(\mathbf{D}_{\overline{\mathbf{X}}})$$

*Proof.* The $S_X$ index takes values in the range

$$S(\mathbf{D}_X) \in \{0, 2, 4, ..., n(n-1)k\}$$

Then, there exists an even integer value $d$ in this range such that $S(\mathbf{D}_X) = d$. Now let us consider a society with 2 individuals that differ in $d/2$ attributes. The diversity matrix associated to this distribution is

$$\mathbf{D}_{\overline{X}} = \begin{pmatrix} 0 & \frac{d}{2} \\ \frac{d}{2} & 0 \end{pmatrix}$$

and the value of the overall diversity index is

$$S(\mathbf{D}_{\overline{\mathbf{X}}}) = d$$

then by transitivity $S(\mathbf{D}_X) = S(\mathbf{D}_{\overline{\mathbf{X}}})$we complete the proof. □

In the following, we denote with $\mathbf{X}^*$ the simple matrix associated to $\mathbf{X}$, built following Lemma 1 and $\overline{\mathbf{X}}$ an other simple matrix related to $\mathbf{X}$ obtained following Lemma 2.

**Remark 2.** *Let* $\mathbf{X}$ *be a simple matrix as defined by Lemma 1,* $\mathbf{X} \in M(5, 1)$*. It is easy to calculate the overall diversity index:* $S(\mathbf{X}) = 8$*. Now we add two individuals identical to the first four individuals, in this case we have* $\mathbf{X}' \in M(7, 1)$ *and* $S(\mathbf{X}') = 12$*. Do not surprise that in the second case we have a greater degree of diversity because now we have a society in which the only individual that has a different attribute value is more emarginate that in the first case. If someone does not agree with this idea because he/she considers the first society must be more diverse than the second one, then it is possible to introduce a normalization in the number of individuals in both the aggregation step. In the following we continuos to consider the second one as more diverse*

Let us introduce a binary relation $\succeq$ as a measure of diversity. For any two societies $\mathbf{X} \in M(n, k)$ and $\mathbf{Y} \in M(n', k')$ the statement $\mathbf{X} \succeq \mathbf{Y}$ means that the degree of diversity offered by $\mathbf{X}$ is greater than or equal to the degree of diversity offered by $\mathbf{Y}$. The relation $\sim$ and $\succ$ derived by $\succeq$ as usually: $\sim$ represents the symmetric part and $\succ$ the asymmetric one.

The diversity relation we introduce is a complete relations because for any two distinct societies $\mathbf{X}$ and $\mathbf{Y}$ it is always possible to assert if

$$\mathbf{X} \succeq \mathbf{Y} \text{ or } \mathbf{X} \preceq \mathbf{Y} \ \ \forall \mathbf{X}, \mathbf{Y} \in \mathbf{M}, \ \mathbf{X} \neq \mathbf{Y}$$

Furthermore our ranking is a reflexive relation

$$\mathbf{X} \succeq \mathbf{X} \ \text{ for all } \ \mathbf{X} \in \mathbf{M}$$

this means that each element of $\mathbf{M}$ is at least as diverse as,or equivalently its degree of diversity is grater than or equal to, itself.

In additional, this relation is also transitive: for any $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \mathbf{M}$

$$\mathbf{X} \succeq \mathbf{Y} \quad \text{and} \quad \mathbf{Y} \succeq \mathbf{Z} \quad \Longrightarrow \quad \mathbf{X} \succeq \mathbf{Z}$$

This means that if the degree of diversity associated to $\mathbf{X}$ is greater than or equal to $\mathbf{Y}$ and $\mathbf{Y}$ in turn has a degree of diversity at least as $\mathbf{Z}$ then $\mathbf{X}$ has a degree of diversity at last as $\mathbf{Z}$.

So we have a complete quasi-ordering.[16]

Finally, we define the counting diversity ordering $\succeq_S$as the ordering represented by the index $S(\mathbf{s}_X)$ formalized in (4.2). For any two societies $\mathbf{X}$ and $\mathbf{Y}$ on the space $\mathbf{M}$, we have that $\mathbf{X} \succeq_S \mathbf{Y}$ if and only if the sum of the elements of the diversity profile $\mathbf{X}$ is greater then or equal to the sum of the elements of the diversity profile $\mathbf{Y}$.

**Definition 3** (Counting diversity ordering)**.** *Let $\mathbf{X} \in M(n, k)$ and $\mathbf{Y} \in M(n', k')$ be two societies and $S$ be a function defined in (2.2) such that*

$$S(\mathbf{D}_X) = \sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij} \quad \textit{and} \quad S(\mathbf{D}_Y) = \sum_{i=1}^{n'} \sum_{j=1}^{n'} d'_{ij}$$

*with $d_{ij} \in \mathbf{D}_X$ and $d'_i \in \mathbf{D}_Y$ then the counting ordering is defined by letting:*

$$\mathbf{X} \succeq_S \mathbf{Y} \quad \Leftrightarrow \quad S(\mathbf{D}_X) \geq S(\mathbf{D}_X)$$

Without losing any generality we can replace $S(\mathbf{D}_X)$ with $S_X$ if it does not cause any ambiguity.

Combining the definition of diversity ordering with the statements of Lemma 1 and Lemma 2, it is easy to check that for all society $\mathbf{X} \in \mathbf{M}$ it is always possible to find two very simple societies, one with several individuals endowed with only an attribute and a second one with only two individuals but with several attributes that are indifferent to $\mathbf{X}$, as asserted with the Lemmas, and then indifferent each other

$$\left.\begin{array}{c} \mathbf{X} \sim_s \mathbf{X}^* \\ \mathbf{X} \sim_s \overline{\mathbf{X}} \end{array}\right\} \Longrightarrow \mathbf{X}^* \sim_s \overline{\mathbf{X}}$$

Now we impose some axioms on the diversity ordering.

[16] A complete quasi-ordering is a reflexive and transitive order that is also complete.

4.1. **Axioms.** In this section we present some axioms in order to characterize our counting diversity ordering.

The first axiom we introduce is the Monotonicity Axiom. It applies to diversity comparison of society in which there are at most two individuals.[17] This axiom requires that the ranking of two societies with only two individuals each, is simply determined by computing the diversity between the two individuals. In other words, let $\mathbf{X}$ and $\mathbf{Y}$ be two matrices associated with two distinct societies with two individuals each $x_1, x_2 \in \mathbf{X}$ and $y_1, y_2 \in \mathbf{Y}$ and let $d_{12} \in \mathbf{D}_X$ and $d'_{12} \in \mathbf{D}_Y$ the degree of diversity between the two individuals in the two societies, then the ranking is obtained only taking in account the value of the degree of diversity between $x_1$ an $x_2$ compared with $y_1$ and $y_2$ ones.

**Axiom 7** (Monotonicity)**.** *For all* $\mathbf{X} \in M(2,k)$ *and* $\mathbf{Y} \in M(2,k')$ *with* $k$ *and* $k'$ *non necessary distinct*

$$\mathbf{X} \succeq \mathbf{Y} \quad \Leftrightarrow \quad d_{12} \geq d'_{12}$$

This means to require that the ranking is determined by the individual Distance between the individuals.

The second axiom we propose is a sort of Separability axiom. It suggests how the ordering works when we incorporate either one individual or several equal individuals in a society. Add groups of individuals are an useful request, for instance in poverty measurement or in inequality. This axiom represents an invariance property with respect to the addition of individuals under some circumstances.

Consider two distinct societies $\mathbf{A} \in M(n,k)$ and $\mathbf{B} \in M(m,k)$ and suppose to incorporate in each societies $\mathbf{A}$ and $\mathbf{B}$ a group of identical individuals $g_A$ and $g_B$ respectively. We denote with $n_A$ the number of individuals in $g_A$ and $n_B$ the number of individuals in $g_B$. The Independence axiom requires that under some circumstance the relative ranking of $\mathbf{A}$ and $\mathbf{B}$ according to $\succeq$ is unchanged if the group $g_A$ is added in $\mathbf{A}$ and $g_B$ in $\mathbf{B}$. The condition we need is that the degree of diversity, when we incorporate, in each group is the same. In other word if $\mathbf{A} \in M(n,k)$ and $|g_A| = n_A$ we denote with $\mathbf{A}' = \mathbf{A} \cup \{g_A\}$ this new matrix and the new diversity matrix is a block matrix of the form:

$$\mathbf{D}_{\mathbf{A}'=\mathbf{A}\cup\{g_A\}} = \begin{pmatrix} \mathbf{D}_\mathbf{A} & \mathbf{G}_\mathbf{A} \\ \mathbf{G}_\mathbf{A}^T & \mathbf{0} \end{pmatrix}$$

where $\mathbf{D}_\mathbf{A}$ is the $(n \times n)$ diversity matrix associated to $\mathbf{A}$, $\mathbf{0}$ is the $(n_A \times n_A)$ null matrix that represents the diversity among individuals in the $g_A$ group and since all individuals are equal the between diversity is null. $\mathbf{G}_\mathbf{A}$ is a $(n \times n_A)$ matrix that which elements represent the degree of diversity of individual in $\mathbf{A}$ respect to the new individuals we added and $\mathbf{G}_\mathbf{A}^T$ is the transpose of $\mathbf{G}_\mathbf{A}$ and represents the diversity between individuals in the group $g_A$ when they are added in society $\mathbf{A}$. It is clear that each column of $\mathbf{G}_\mathbf{A}$ are equal and each row of $\mathbf{G}_\mathbf{A}^T$ too. In addition, by symmetry the these two block matrices are identical: $(\mathbf{G}_\mathbf{A})^T = \mathbf{G}_\mathbf{A}^T$. Let $s_i^{A'}$ be the sum, by row of the elements of the $\mathbf{G}_\mathbf{A}^T$ matrix, in other words the individual diversity score in $\mathbf{A}'$ for any individual $i \in g_A$. Then the overall diversity index of the group $g_A$ in $\mathbf{A}'$ is $n_A$ times the individual diversity score: $n_A s_i^{A'}$. We apply the same procedure for society $\mathbf{B}$ and group $g_B$. With the same notation we call $n_B s_j^{B'}$ the overall diversity index of group $g_B$ added in $\mathbf{B}$ where $s_j^{B'}$ is the score in $\mathbf{B}'$ for individual $j \in g_B$.

Then our condition is

$$n_A s_i^{A'} = n_B s_j^{B'}$$

[17]We can apply to society with the same number of attributes or with a different one.

If this requirement is satisfied, our Separability axiom affirms that the relative ranking of society for society $\mathbf{A}$ and $\mathbf{B}$ to be the same as the relative ranking of $\mathbf{A'} = \mathbf{A} \cup \{g_A\}$ and $\mathbf{B'} = \mathbf{B} \cup \{g_B\}$. In other words, our idea is that if we add simultaneously groups of individuals to the two societies with the restrictions introduced above do not reverse the ranking..

**Axiom 8** (Separability)**.** *For all any* $\mathbf{A} \in M(n,k)$ *and* $\mathbf{B} \in M(m,k)$ *for any two groups* $g_A$ *and* $g_B$ *of* $n_A$ *and* $n_B$ *identical individuals respectively, if* $n_A s_i^{A'} = n_B s_j^{B'}$ *then*

$$\mathbf{A} \succeq \mathbf{B} \quad \Leftrightarrow \mathbf{A} \cup \{g_A\} \succeq \mathbf{B} \cup \{g_B\}$$

These two axioms provide a characterization of the Counting Diversity Ordering.

We denote the overall diversity index for a society of two individuals with $d(y_1, y_2)$ if this does not confuse.

4.2. **Characterization of the ordering.** The Separability Axiom we introduced in the previous section allows us to we assert that it is possible to find two societies that are indifferent and then shows a rule to compare these two simple societies. It affirms that, given a society $\mathbf{A} \in M(2,k)$ it is possible to associate according Lemma 1 a simple matrix $\mathbf{A}^* \in M(\frac{s_A}{2} + 1, 1)$ such that these two societies are indifferent.

**Lemma 3.** *For all any* $\mathbf{A} \in M(2,k)$ *and* $\mathbf{A}^*$ *the simple society built according Lemma 1,* $\mathbf{A}^* \in M(\frac{s_A}{2} + 1, 1)$ *and for any diversity ordering fulfilling the Separability Axiom, then*

$$\mathbf{A} \sim \mathbf{A}^*$$

*Proof.* We prove that the Separability Axiom implies that for all non negative integer $d_{xy}$

$$\begin{pmatrix} 0 & d_{xy} \\ d_{xy} & 0 \end{pmatrix} \sim \underbrace{\begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}}_{d_{xy}+1}$$

Let

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

In the first society we add one individual whose diversity score is $2d_{xy}$. We observe that an individual with this degree of diversity always exists: we need to assume that in this society individual are compared among $d_{xy}$ and this third individual we add differs from individual $x$ and $y$ in all the attributes.

$$\begin{pmatrix} 0 & 1 & d_{xy} \\ 1 & 0 & d_{xy} \\ d_{xy} & d_{xy} & 0 \end{pmatrix}$$

In the second society we add $2d_{xy}$ individuals, all identical among them, and identical to one of two individuals already existing in the society.

$$\underbrace{\begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}}_{2d_{xy}+2}$$

Then, by separability Axiom, we get

$$\begin{pmatrix} 0 & 1 & d_{xy} \\ 1 & 0 & d_{xy} \\ d_{xy} & d_{xy} & 0 \end{pmatrix} \sim \underbrace{\begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}}_{2d_{xy}+2}$$

Now we can remove the second individual in the fist society

$$\begin{pmatrix} 0 & d_{xy} \\ d_{xy} & 0 \end{pmatrix}$$

and the $d_{xy} + 1$ individuals in the second society

$$\underbrace{\begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}}_{d_{xy}+1}$$

and applying again the Separability Axiom we get the result. □

Then we can formulate our main theorem.

**Theorem 1.** *A diversity ordering $\succeq$ satisfies Monotonicity Axiom and Separability Axiom if and only if $\succeq = \succeq_S$.*

*Proof.* It is straightforward to prove that $\succeq_S$satisfies the required axioms.

Now we suppose that $\succeq$ satisfies Monotonicity Axiom and Separability Axiom.

We firstly prove that $\forall \mathbf{A} \in M(n+1, k)$ and $\mathbf{A}^*$ built according to Lemma 1, then

$$S(\mathbf{D}_A) = S(\mathbf{D}_{A^*}) \Rightarrow \mathbf{A} \sim \mathbf{A}^* \tag{4.4}$$

In order to prove this implication we proceed by induction on the number of individuals.

If $\mathbf{X} \in M(2, k)$ then equation (4.4) derives by Lemma 3.

Now suppose that equation (4.4) holds for all positive integer $n$ (induction hypothesis) then we show that this implies equation (4.4) for all society $\mathbf{A}$ with $n+1$ individuals.

We can write the overall diversity index $S(\mathbf{D}_A)$ as follows

$$S(\mathbf{D}_A) = \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} d_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij} + 2 \sum_{j=1}^{n+1} d_{j,n+1}$$

We denote by $\mathbf{A}\setminus\{\mathbf{x}_{n+1}\}$ the society $\mathbf{A}$ in which we remove the last individual and we observe that $\sum_{j=1}^{n+1} d_{j,n+1}$ is the individual diversity for individual $n+1$ in the society $\mathbf{A}$

$$\sum_{j=1}^{n+1} d_{j,n+1} = s_{n+1}^{A}$$

In this way we rewrite the previous equation as

$$\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij} + 2\sum_{j=1}^{n+1} d_{j,n+1} = S(\mathbf{D}_{A\setminus\{\mathbf{x}_{n+1}\}}) + 2s_{n+1}^{A}$$

Using Lemma 1, it is always possible to find a simple society

$$S(\mathbf{D}_{A\setminus\{\mathbf{x}_{n+1}\}}) = S(\mathbf{D}_{(A\setminus\{\mathbf{x}_{n+1}\})^*})$$

and by induction hypothesis we have

$$\mathbf{A}\setminus\{\mathbf{x}_{n+1}\} \sim (\mathbf{A}\setminus\{\mathbf{x}_{n+1}\})^*$$

Now we suppose to add in $\mathbf{A}\setminus\{\mathbf{x}_{n+1}\}$ an individual $\mathbf{x}_{n+1}$ and in the simple society $(\mathbf{A}\setminus\{\mathbf{x}_{n+1}\})^*$ to add a group $g_A$ of $s_{n+1}^{A}$ identical individuals and identical to the identical individuals in $(\mathbf{A}\setminus\{\mathbf{x}_{n+1}\})^*$

$$\begin{gathered}(\mathbf{A}\setminus\{\mathbf{x}_{n+1}\}) \cup \{\mathbf{x}_{n+1}\} = \mathbf{A}\\ (\mathbf{A}\setminus\{\mathbf{x}_{n+1}\})^*) \cup \{g_A\} = \mathbf{A}^*\end{gathered}$$

then, in $\mathbf{A}$ we the degree of diversity of this new individual we add is $1 \cdot s_{n+1}^{A}$ and in the simple society we have exactly $s_{n+1}^{A}$ individuals with the same degree of diversity (1) then the degree of diversity is $s_{n+1}^{A} \cdot 1$.

Then by Separability Axiom we have

$$\mathbf{A} \sim \mathbf{A}^*$$

that completes the first part of our theorem.

Now let $\mathbf{X} \in M(n,k)$ and $\mathbf{Y} \in M(n',k')$. By Lemma 1 it is possible to find $\mathbf{X}^*$ and $\mathbf{Y}^*$ and by equation (4.4)

$$\mathbf{X} \sim \mathbf{X}^* \quad \text{and} \quad \mathbf{Y} \sim \mathbf{Y}^* \tag{4.5}$$

where

$$\mathbf{D}_{\mathbf{X}^*} = \underbrace{\begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}}_{\frac{S(\mathbf{D}_X)}{2}+1}$$

By Lemma 2 we can construct $\mathbf{D}_{\overline{\mathbf{X}}}$.where

$$\mathbf{D}_{\overline{\mathbf{X}}} = \begin{pmatrix} 0 & \frac{S(\mathbf{D}_X)}{2} \\ \frac{S(\mathbf{D}_X)}{2} & 0 \end{pmatrix}$$

We also get $\mathbf{D}_{(\overline{\mathbf{X}})^*}$ where

$$\mathbf{D}_{(\overline{\mathbf{X}})^*} = \underbrace{\begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}}_{\frac{S(\mathbf{D}_X)}{2}+1}$$

and by equation (4.5) we have

$$\overline{\mathbf{X}} = (\overline{\mathbf{X}})^* \tag{4.6}$$

we realize that $\mathbf{D}_{\mathbf{X}^*} = \mathbf{D}_{(\overline{\mathbf{X}})^*}$.Then

$$(\overline{\mathbf{X}})^* \sim \mathbf{X}^* \quad \text{and} \quad \mathbf{X}^* \sim \mathbf{X} \tag{4.7}$$

and using transitivity by equations (4.7) and equation (4.6) we obtain $\mathbf{X} \sim \overline{\mathbf{X}}$ and similarity for $\mathbf{Y} : \mathbf{Y} \sim \overline{\mathbf{Y}}$ where

$$\overline{\mathbf{Y}} = \begin{pmatrix} 0 & \frac{S(\mathbf{D}_Y)}{2} \\ \frac{S(\mathbf{D}_Y)}{2} & 0 \end{pmatrix}$$

and by Monotonicity

$$\overline{\mathbf{X}} \succeq \overline{\mathbf{Y}} \Leftrightarrow \frac{S(\mathbf{D}_X)}{2} \geq \frac{S(\mathbf{D}_Y)}{2}$$

that completes the proof. □

## 5. Conclusions

There is a growing interest in defining and measuring diversity. The primary goal of this work was to present and characterize a diversity index and a counting diversity ordering based on a diversity index that takes in account the number of attributes in which individuals in a society differ. So doing we introduced a distance that allows us to capture this degree of diversity simply by counting the number of attributes in which individuals differ. The use of this distance consents to be able to evaluate the variation in the diversity degree over time in order to help a policy maker in defining strategies that reduce diversity among individuals if we think to diversity as an iniquitous concept

Our approach finds natural applications in different contexts. For instance in poverty measurement or in a deprivation context it is possible to use this kind of distance that respectively counts the number of attributes in which people belongs to a poverty line or the number of dimensions in which individuals are deprived.

Moreover, it is possible to apply this distance also in non-economic contexts, for instance in information theory. In this case the distance we use is the so called Hamming Distance, a distance introduced in 1950 by Richard Hamming. As we stressed above this distance is a metric and in that context it counts the number of letters in which two different code-words of the same length differ. Thus we can affirm that the Hamming Distance well interpret the counting approach.

This counting approach is an adequate procedure also in diversity context because it allows us to work with ordinal and categorical variables.

Feature works are addressed in several directions: firstly we would to found dominance conditions in order to guarantee unanimous diversity counting ranking in a counting framework if profiles of diversity of different societies do not intersect. Finally we would to investigate the relationship among diversity, poverty, deprivation and social exclusion: our purpose is to tie the analysis of poverty, in a multidimensional point of view, to the diversity context.

## References

[1] Alkire S., Foster, J. *Counting and Multidimensional Poverty Measurement*, Journal of Public Economics, **95,**7-8 (2011), 476-487.

[2] Atkinson A.B., *Multidimensional deprivation: Contrasting social welfare and counting approach*, Journal of Economic Inequality **1** (2003), 51-65.

[3] Barberà S., Bossert W., Pattanaik P.K., *Ranking sets of objects,* in Barberà S., Hammond P.C., Seidl C. (eds): Hanbook of Utility Theory, vol II, Extensions, Kluwer of options and the measurement of diversity, Academic Publishers, Dordrecht (2004), 893-977.

[4] Bervoets S., Gravel N., *Appraising diversity with an ordinal notion of Similarity: an Axiomatic Approach,* Mathematical Social Sciences, **53**, 3 (2007), 259-273.

[5] Bossert, W., S.R. Chakravarty and C. D'Ambrosio, *Measuring Multidimensional Poverty: the Generalized Counting Approach*, ECINEQ Working Paper, 2009-129.

[6] Bossert W., D'Ambrosio C., Peragine V., *Deprivation and Social Exclusion,* Economica, **74** (2007), 777-803.

[7] Bossert W., Pattanaik P.K., Xu Y., *Similarity of option and the measurement of diversity,* Journal of Theoretical Politics, (2003), 405-421.

[8] Chakravarty, S.R., D'Ambrosio, C, *The measurement of social exclusion,* Review of Income and Wealth, **52,** 3 (2006), 377-398.

[9] Dalton, H., *The Measurement of the Inequality of Incomes,* Economic Journal, **30**,119 (1920), 348-361.

[10] Dutta, I., Pattanaik, P.K., Xu, Y., *On measuring Deprivation and the Standard of Living in a Multidimensional Framework on the Basis of Aggregate Data,* Economica **70** (2003), 197-221.

[11] Lasso de la Vega, M.C, *Counting poverty orderings and deprivation curves*, Studies in Applied Welfare Analysis, Research on Economic Inequality, **18** (2010), 153-172.

[12] Nehring, K., Puppe, C., *A Theory of Diversity,* Econometrica, **70** (2002), 1155-1190.

[13] Nehring, K., Puppe, C., *Diversity,* The Handbook of Rational and Social Choice, ed. by P. Anand, P. Pattanaik and C. Puppe: Oxford University Press (2009), Chapter 12.

[14] Pattanaik P.K., Xu, Y. *On diversity as freedom of choice*, Mathematical Social Sciences, **40** (2002), 123-130.

[15] Pattanaik P.K., Xu, Y, *Ordinal Distance, Dominance, and the Measurement of Diversity,* Working Paper, May 2006.

[16] Sen, A., *Poverty: An Ordinal Approach to Measurement*, Econometrica, **44**, 2 (1976), 219-231.

[17] Shannon, C.E., *A Mathematical Theory of Communication*, Bell System Technical Journal, **27** (1948), 623–656.

[18] Sudgen, R. *The Metric of Opportunity*", Economics and Philosophy, **14** (1998), 307-337.

[19] Weitzman M. L. , *On diversity*, The Quarterly Journal of Economics, **107**, 2 (1992), 363-405.

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