

## OPTIMAL DYNAMIC TAX EVASION: A PORTFOLIO APPROACH

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# Optimal dynamic tax evasion: a portfolio approach

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## Abstract

Most tax evasion models are set in a timeless environment and assume that only income flow can be evaded. This framework is not suitable for financial market where an evasion decision is taken in an intertemporal framework and an asset itself can be evaded. We assume that a representative agent may invest on a risky asset (following a geometric Brownian motion) and on a riskless asset. The risky asset can be either declared for taxation or not. If it is not declared a fine must be paid with a given probability. In our framework the agent optimally chooses his intertemporal consumption and portfolio allocation where the "declared risky asset" and the "undeclared (evaded) risky asset" are treated as different assets. The main results are: (i) optimal consumption is higher with evasion, (ii) optimal evasion is affected neither by the return nor by the volatility of the risky asset, (iii) evasion reduces the investment in the risky asset and increases the investment in the riskless asset, (iv) evasion can be reduced more efficiently by increasing the amount of fine rather than increasing the frequency of controls, (v) for level of tax on the riskless asset sufficiently high tax evasion is not optimal.

**JEL Classification:** G11, H26,H42

**Key Words:** dynamic tax evasion; asset allocation;

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# 1 Introduction

Tax evasion is probably one of the most studied and less desired effects of Government intervention in the economy. Since the seminal papers by Allingham and Sandmo (1972) and Yitzaki (1974), the literature on tax evasion has been offering explanations and possible cures for this phenomenon. In spite of this great effort, tax evasion seems to increase; Schneider (2003, 2005) shows that the shadow economy, a good proxy for tax evasion, has been increasing in OECD and transition economies (from 13.2% in 1990 to 16.7% in 2001 in OECD countries). For US, the most recent estimates (Cebula and Feige, 2011) show that intentional underreporting of income is about 18-19% of total reported income giving rise to a tax gap of about \$500 billion dollars. Tax evasion produces pervasive effects on economic growth, on the distribution of the tax burden, and on the relative cost of public sector activities (Levaggi, 2007). Even though the literature does not fully agree on the desirability of reducing tax evasion (Davidson and Wilson, 2007), the common trait of most of the analyses proposed is that the level of tax evasion is decided in a timeless environment where the decision to evade and tax audit are made at the same time. However, tax evasion is a dynamic decision, especially if it is correlated to systematic income underreporting. Auditing is an intertemporal process: detection triggers investigation on prior and, possibly, future reporting (Allingham and Sandmo, 1972; Engel and Hines, 1999) and income that is evaded may itself produce revenue that an agent can decide to report or not. These considerations are highly relevant for financial assets for which risk and intertemporal decisions are important dimensions.

Despite the importance of the intertemporal dimension, only few attempts have been made in that direction. Some authors try to investigate the relationship between tax rate, tax evasion, and economic growth (Lin and Yang, 2001; Dalamagas, 2011; Dzhumashev and Gaharamanov, 2011). These models study several aspects related to income underreporting in a framework where consumers are concealing a part of their income flow. Nevertheless, we argue that such a framework is not suitable to study tax evasion on financial markets where agents conceal assets and their income flows. Some financial activities, with a particular risk-return profile, cannot be evaded by their nature.

Niepelt (2005) examines the problem tax evasion in a true dynamic framework and shows the optimal path of tax evasion for a risk neutral agent. The most important finding of this paper is that an interior solution exists: the individual will choose to evade a part of its income rather than one of the two corner solutions.

The model proposed in this paper studies the decision of capital tax evasion in an optimal portfolio allocation framework. We solve the problem of a risk averse agent who intertemporally optimizes portfolio allocation and his utility of consumption. The decision about asset allocation is made on three financial assets: the riskless asset (on which investor cannot evade), the "declared" risky asset, and the "undeclared" risky asset. The "declared" and "undeclared" risky assets follow different stochastic processes. In fact, in the event of a tax audit,

the investor must pay a fine proportional to the value of the assets that have been concealed (with a given probability that depends on the frequency of the audits).

The paper offers several contributions to the existing literature on tax evasion and portfolio allocation. In particular, it shows that: (i) optimal consumption is higher with tax evasion, (ii) the optimal level of tax evasion is affected neither by the return nor by the volatility of the risky asset, (iii) evasion reduces the investment in the risky asset and increases the investment in the riskless asset, (iv) evasion can be reduced more efficiently by increasing the amount of the fine rather than increasing the frequency of controls, (v) evasion is increasing in the tax rate of the asset that can be concealed, but it decreases in the tax rate on the riskless asset.

The paper is structured as follows. In Section 2 the model is presented. In Section 3 the optimal consumption and portfolio without evasion are computed as a benchmark case. Section 4 presents the optimal consumption and portfolio with evasion and, thus, the main results of the paper. Section 5 summarizes some policy implication for the Government. Section 6 concludes. Some technical results are stored in an appendix.

## 2 The model

We take into account a frictionless financial market in continuous time where two assets are listed:

1. a riskless asset whose (constant) return is  $r$  and whose price  $G(t)$  solves the (deterministic) differential equation

$$\frac{dG(t)}{G(t)} = rdt,$$

with an initial value in  $t_0$  given by  $G(t_0) = G_0 > 0$ . We can think of this investment as a Government bond or as liquidity on a bank/deposit account. Income on this riskless asset cannot be evaded;

2. a risky asset whose (constant) expected return is  $\mu$  and whose price  $S(t)$  follows a geometric Brownian motion

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW(t),$$

where  $\sigma$  measures the standard deviation of risky return and  $dW(t)$  is a Wiener process (whose normal density has zero mean and  $dt$  variance). This asset can be declared to the tax authority, in which case its return is taxed, or it may be concealed (and its return is not taxed). Here, we assume  $\mu > r$  (even if this hypothesis is not necessary for having a complete and arbitrage free financial market, it is more in line with the empirical data).

This financial market is arbitrage free and complete. In fact there exists a unique market price of risk which coincides with the Sharpe ratio  $\frac{\mu-r}{\sigma}$ . The assets we model do not pay coupons/dividends. For investors, the gain coincides with the accrual in the asset values.

## 2.1 The tax system

The taxation of financial activities is one of the most complicated part of any tax system. The tax rate usually depends on the income source, on the type of investor, on time horizon, on the objective of the investment itself, and on the rules to determine the tax base (accrued or realized capital gain). Poterba (2002) and Bergstrasser and Poterba (2004) discuss this point and show the effects of the tax system on the optimal portfolio allocation. In our model we have tried to capture the features that are most relevant in a dynamic setting. We assume that Government imposes taxes on invested income but not on its use, i.e. consumption is not taxed. The revenue is taxed in a symmetric way through capital income tax based on the accrual, i.e. the tax base is represented by the change in the asset value. The tax is paid if the latter is positive, while the investor receives a refund if the change in the asset value is negative (in other words, a loss on the investment allows the investor to have a refund).

The tax rate on assets does not need to be uniform; to keep the model as general as possible, we allow for different tax rates between assets:

1. a tax  $\tau_G \in [0, 1]$  is levied on the riskless payoff  $dG(t)$ ; the net payoff is then

$$w_G(t) (1 - \tau_G) dG(t),$$

where  $w_G(t)$  is the number of riskless asset held in portfolio at time  $t$ ;

2. a tax  $\tau \in [0, 1]$  is levied on the risky payoff  $dS(t)$ ; the net payoff is then

$$w(t) (1 - \tau) dS(t),$$

where  $w(t)$  is the number of risky asset held in portfolio at time  $t$ .

Tax audits are performed with an intensity  $\lambda$  which determines the probability of the audit itself. In the case of audit any income that has been concealed from the tax authority is detected and the investor has to pay a fine  $\theta \in [0, 1]$  levied on the total value of the evaded assets (as in Allingham and Sandmo, 1972).

In our model the auditing process has no memory: in other words the probability of being audited depends neither on the number nor on the result of the audits undergone.

The riskless investment cannot be concealed to the tax authority. Instead, we assume that the investor can hide a part of the wealth invested in the risky asset and the income flow derived from such wealth. There are no costs associated with concealing or emerging assets. This hypothesis does not seem to be strong since financial capital can be more easily concealed than other income sources.

We have assumed zero cost to emerge capital for symmetry; this assumption is not relevant in our model since the optimal tax evasion is a constant proportion of wealth whose expected value is constantly increasing.

The investor can then choose to hide a number  $w_0(t)$  of risky assets whose payoff is given by

$$w_0(t) dS_0(t).$$

In the event of an audit a fine on the payoff  $dS_0(t)$  will have to be paid. This alters the expected return of this asset. The payoff  $dS(t)$ , we must be reduced by the amount of the fine  $\theta$ , weighted by a stochastic process measuring the frequency of audits. Thus  $dS_0(t)$  can be modelled as

$$\frac{dS_0(t)}{S_0(t)} = \mu dt + \sigma dW(t) - \theta d\Pi(t),$$

where  $d\Pi(t)$  is a jump Poisson process whose (constant) intensity is  $\lambda$  (and thus the expected value of  $d\Pi(t)$  is  $\lambda dt$ ). When the investor has to pay the percentage fine  $\theta$ , the amount of wealth invested in  $S_0(t)$  falls by the amount  $S_0(t)\theta$ . We assume that the stochastic process  $d\Pi(t)$  is independent on  $dW(t)$ . This means that the frequency of controls does not depend on the financial risk on the risky asset.

The solution of the differential equation for  $S_0(t)$  can be found by applying Itô's lemma to  $\ln S_0(t)$  as follows:

$$d \ln S_0(t) = \left( \mu - \frac{1}{2} \sigma^2 \right) dt + \sigma dW(t) + \ln(1 - \theta) d\Pi(t),$$

where we see that this solution exists if and only if  $\theta < 1$ . In other words, our model makes sense if and only if the fee is lower than the value of the assets that have been concealed for tax purposes.

## 2.2 The investor's choice

The representative investor wants to maximise the intertemporal utility of his consumption  $c(t)$  and his final wealth  $R(T)$ , where  $T$  is his time horizon. Investor's preferences belong to the Constant Relative Risk Aversion family (CRRA), i.e. the utility of consumption is given by

$$U(c(t)) = \frac{c(t)^{1-\delta}}{1-\delta},$$

where  $\delta$  is the (constant) Arrow-Pratt relative risk aversion index. In order to make the problem consistent, we will assume  $\delta \geq 1$ .<sup>1</sup> Utility is discounted at a subjective constant discount rate  $\rho$ . The intertemporal optimization problem can be written as:

$$\max_{w(t), w_0(t), c(t)} \mathbb{E}_{t_0} \left[ \int_{t_0}^T \frac{c(t)^{1-\delta}}{1-\delta} e^{-\rho(t-t_0)} dt + \chi \frac{R(T)^{1-\delta}}{1-\delta} e^{-\rho(T-t_0)} \right], \quad (1)$$

<sup>1</sup>Please note that when  $\delta = 1$  the investor behaves as he had a log-utility.

where  $\mathbb{E}_{t_0}$  is the expected value operator (conditional on information at time  $t_0$ ), and the final wealth is weighted by  $\chi \geq 0$ . The higher  $\chi$  the stronger the investor's preference towards final wealth with respect to intertemporal consumption.

Investor's wealth  $R(t)$  must be constantly equal to his portfolio value, i.e.

$$R(t) = w_G(t) G(t) + w(t) S(t) + w_0(t) S_0(t).$$

Under the usual self-financing condition, the dynamics of this constraint is

$$dR(t) = w_G(t) (1 - \tau_G) dG(t) + w(t) (1 - \tau) dS(t) + w_0(t) dS_0(t) - c(t) dt,$$

where  $c(t)$  is the amount of wealth consumed.

Substituting  $w_G(t)$  from the static budget constraint into the dynamic budget constraint, we have

$$\begin{aligned} dR(t) &= (R(t) - w(t) S(t) - w_0(t) S_0(t)) (1 - \tau_G) r dt \\ &\quad + w(t) (1 - \tau) dS(t) + w_0(t) dS_0(t) - c(t) dt, \end{aligned}$$

and, finally,

$$\begin{aligned} dR(t) &= \left( \begin{aligned} &R(t) (1 - \tau_G) r + w(t) S(t) ((1 - \tau) \mu - (1 - \tau_G) r) \\ &+ w_0(t) S_0(t) (\mu - (1 - \tau_G) r) - c(t) \end{aligned} \right) dt \quad (2) \\ &\quad + (w(t) (1 - \tau) S(t) + w_0(t) S_0(t)) \sigma dW(t) - w_0(t) S_0(t) \theta d\Pi(t). \end{aligned}$$

### 3 Optimal tax evasion and portfolio allocation

**Proposition 1** *The optimal consumption and asset allocation solving problem (1), given the wealth differential (2), are:*

$$\frac{c^*(t)}{R(t)} = \frac{1}{\frac{1 - e^{-\phi(T-t)}}{\phi} + \chi^{\frac{1}{\delta}} e^{-\phi(T-t)}},$$

$$\frac{w_0^*(t) S_0(t)}{R(t)} = \frac{1}{\theta} \left( 1 - \left( \frac{(1 - \tau) \theta \lambda}{(1 - \tau_G) r \tau} \right)^{\frac{1}{\delta}} \right), \quad (3)$$

$$\frac{w^*(t) S(t)}{R(t)} = \frac{1}{\delta} \frac{(1 - \tau) \mu - (1 - \tau_G) r}{(1 - \tau)^2 \sigma^2} - \frac{1}{1 - \tau} \frac{w_0^*(t) S_0(t)}{R(t)}, \quad (4)$$

$$\frac{w_G^*(t) G(t)}{R(t)} = 1 - \frac{1}{\delta} \frac{(1 - \tau) \mu - (1 - \tau_G) r}{(1 - \tau)^2 \sigma^2} + \frac{\tau}{1 - \tau} \frac{w_0^*(t) S_0(t)}{R(t)}, \quad (5)$$

where

$$\begin{aligned} \phi &\equiv \frac{\rho + \lambda}{\delta} + \frac{\delta - 1}{\delta} (1 - \tau_G) r + \frac{1}{2} \frac{\delta - 1}{\delta^2} \left( \frac{(1 - \tau) \mu - (1 - \tau_G) r}{(1 - \tau) \sigma} \right)^2 \\ &\quad + \frac{\delta - 1}{\delta} \lambda \left( \frac{(1 - \tau_G) r \tau}{(1 - \tau) \theta \lambda} \right) - \lambda \left( \frac{(1 - \tau_G) r \tau}{(1 - \tau) \theta \lambda} \right)^{1 - \frac{1}{\delta}} < 1. \end{aligned}$$

**Proof.** See Appendix A. ■

Tax evasion is convenient from the point of view of the investor (i.e.  $\frac{w_0^*(t)S_0(t)}{R(t)} > 0$ ) if

$$\frac{(1-\tau)\theta\lambda}{(1-\tau_G)r\tau} < 1. \quad (6)$$

Before presenting comments on these results, we show that there exist levels of  $\theta$  and  $\lambda$  such that the evasion is zero. In particular, we have

$$\frac{w_0^*(t)S_0(t)}{R(t)} = 0 \iff \frac{1}{\theta} \left( 1 - \left( \frac{(1-\tau)\theta\lambda}{(1-\tau_G)r\tau} \right)^{\frac{1}{\delta}} \right) = 0,$$

from which we have

$$(\lambda\theta)^* = \frac{(1-\tau_G)r\tau}{(1-\tau)}.$$

Please note that the level of  $\lambda\theta$  for which evasion is not convenient depends neither on the risk aversion parameter  $\delta$ , nor on the risky asset risk/return profile (i.e. parameters  $\mu$  and  $\sigma$ ). By using  $(\lambda\theta)^*$  we can easily obtain the result for an agent who cannot evade.

**Corollary 2** *The optimal consumption and asset allocation solving problem (1), given the wealth differential (2), for an agent who does not evade are:*

$$\frac{\hat{c}^*(t)}{\hat{R}(t)} = \frac{1}{\frac{1-e^{-\hat{\phi}(T-t)}}{\hat{\phi}} + \chi^{\frac{1}{\delta}} e^{-\hat{\phi}(T-t)}}, \quad (7)$$

$$\frac{\hat{w}^*(t)S(t)}{\hat{R}(t)} = \frac{1}{\delta} \frac{(1-\tau)\mu - (1-\tau_G)r}{(1-\tau)^2\sigma^2}, \quad (8)$$

$$\frac{\hat{w}_G^*(t)G(t)}{\hat{R}(t)} = 1 - \frac{1}{\delta} \frac{(1-\tau)\mu - (1-\tau_G)r}{(1-\tau)^2\sigma^2}, \quad (9)$$

where

$$\hat{\phi} \equiv \frac{\rho}{\delta} + \frac{\delta-1}{\delta} (1-\tau_G)r + \frac{1}{2} \frac{\delta-1}{\delta^2} \left( \frac{(1-\tau)\mu - (1-\tau_G)r}{(1-\tau)\sigma} \right)^2.$$

**Proof.** In the results of Proposition 1, it is sufficient to substitute  $\lambda\theta = \frac{(1-\tau_G)r\tau}{(1-\tau)}$ . ■

The optimal amount of consumption is given by the inverse of an annuity which gives 1 monetary unit at any instant from  $t$  up to  $T$ ,  $\chi^{\frac{1}{\delta}}$  monetary units in  $T$ , and whose discount rate is  $\phi$  (or  $\hat{\phi}$  without evasion). For the log-investor (with  $\delta = 1$ ) the discount rate is equal to the subjective discount rate  $\rho$ . Instead, for an infinitely risk averse agent (i.e.  $\delta \rightarrow \infty$ ), the discount rate is equal to the net riskless return  $(1-\tau_G)r$ .

The optimal consumption (as a percentage of wealth) may increase or decrease through time according to the value of  $\chi$ . In fact, we have

$$\frac{\partial \left( \frac{c^*(t)}{R(t)} \right)}{\partial t} \geq 0 \iff \chi \leq \phi^{-\delta}.$$



The intuition behind this result is very strong indeed: if  $\chi$  is high (i.e. higher than  $\phi^{-\delta}$ ), then the agent gives a strong importance to the utility of his final wealth and he will try to keep consumption as low as possible (and decrease it through time) in order to save the highest amount of final wealth. Instead, if  $\chi$  is low (i.e. lower than  $\phi^{-\delta}$ ), then the agent's utility mainly depends on the level of intertemporal consumption and he will try to consume as much as he can (by increasing consumption through time).

The level of risk aversion determines the value of  $\phi^{-\delta}$ . In particular, for an infinitely risk averse agent,  $\phi^{-\delta}$  approaches infinity and, accordingly, consumption is increasing through time.

The (percentage) amount of wealth invested in the risky asset is proportional to the Sharpe ratio (whose elements are taken net of taxation) and to the relative risk tolerance index ( $\frac{1}{\delta}$ ). The residual wealth is of course invested in the riskless asset. An infinitely risk averse agent (with  $\delta \rightarrow \infty$ ) would of course invest all his money in the riskless asset and would not evade.

It is worth noting that a uniform taxation on all the assets (i.e.  $\tau_G = \tau$ ) does not alter the market price of risk but does affect the optimal asset allocation. In fact, in the case without evasion, the wealth optimally invested in the risky asset is

$$\left. \frac{\hat{w}^*(t) S(t)}{\hat{R}(t)} \right|_{\tau_G = \tau} = \frac{1}{\delta} \frac{\mu - r}{\sigma^2} \frac{1}{1 - \tau},$$

which is higher than the wealth invested in the risky asset without taxation:

$$\left. \frac{\hat{w}^*(t) S(t)}{\hat{R}(t)} \right|_{\tau_G = \tau = 0} = \frac{1}{\delta} \frac{\mu - r}{\sigma^2}.$$

This is due to the taxation mechanism: since Government participates to both positive and negative returns, then the risk of investing on the asset  $S(t)$  is reduced (i.e. it is shared with the Government) and the investor can allow to invest more money in it.

**Corollary 3** *Evasion (3), which is never negative, reduces the investment in the risky asset:*

$$\frac{\hat{w}^*(t) S(t)}{\hat{R}(t)} > \frac{w^*(t) S(t)}{R(t)},$$

*and increases the investment in the riskless asset:*

$$\frac{\hat{w}_G^*(t) G(t)}{\hat{R}(t)} < \frac{w_G^*(t) G(t)}{R(t)}.$$

**Proof.** The proof directly comes from comparison between the couple (8)-(4) and the couple (9)-(5). ■

This result has important implications: it may account for the observation that the portfolio of individuals is often more "liquid" than it would optimally be. The literature has long tried to explain this phenomenon. The psychological

expected utility theory (Caplin and Leahy, 2001) argue that it may depend on anticipatory feelings on the consequences of losing part of the wealth on risky investment. In our model we show that this choice is fully rational. From a policy point of view, a high liquidity may be interpreted as a signal of tax evasion and it may be used for targeting the audits.

Tax evasion causes a distortion in the optimal asset allocation since it increases the share of total wealth (defined as the sum of wealth declared and not declared) held as liquidity beyond its optimal level: in other words the portfolio of a tax evasor is biased towards the riskless asset.

**Corollary 4** *Consumption with evasion is always greater than consumption without evasion:*

$$\frac{c^*(t)}{R(t)} \geq \frac{\hat{c}^*(t)}{R(t)}.$$

**Proof.** The optimal consumptions  $c^*(t)$  and  $\hat{c}^*(t)$  have exactly the same structure, but a different discount rate  $\phi$ . Accordingly, we must determine whether  $\phi \geq \hat{\phi}$ . We can immediately check that

$$\phi \geq \hat{\phi} \iff \frac{1}{\delta} + \frac{\delta - 1}{\delta} \frac{(1 - \tau_G) r \tau}{(1 - \tau) \theta \lambda} - \left( \frac{(1 - \tau_G) r \tau}{(1 - \tau) \theta \lambda} \right)^{1 - \frac{1}{\delta}} \geq 0.$$

Now, we have to study the function

$$f(x) \equiv \frac{1}{\delta} + \frac{\delta - 1}{\delta} x - x^{1 - \frac{1}{\delta}},$$

where

$$x \equiv \frac{(1 - \tau_G) r \tau}{(1 - \tau) \theta \lambda} > 1,$$

where the inequality holds if evasion is convenient. Since it is easy to show that

$$\begin{aligned} f(1) &= 0, \\ \frac{\partial f(x)}{\partial x} &> 0, \end{aligned}$$

then we can conclude that

$$\phi \geq \hat{\phi}.$$

Now, since  $\phi$  is a discount rate, the higher  $\phi$  the lower the value of the following annuity

$$\int_t^T e^{-\phi(s-t)} ds + \chi^{\frac{1}{\delta}} e^{-\phi(T-t)},$$

and, accordingly, the higher the optimal consumption. ■

This implies that the income effect caused by tax evasion outweighs the substitution effect. In fact, tax evasion increases the expected total income of the investor, but at the same time it increases the relative price of consumption. This increase in consumption means that the effect of tax evasion on total

wealth is uncertain. The increase in consumption means that less wealth will be invested in financial assets which in turn reduces the amount of total wealth. On the other hand, tax evasion increases the expected net return of assets which in turn may increase investment.

The uncertainty on the effect on total wealth means that it is not possible to determine the impact of tax evasion on economic growth.

Finally, it is important to note that the optimal asset allocation is a constant percentage of wealth (and so is evasion). This result is due to the hypothesis that all the parameters are constant. In other words, the optimal allocation between assets, both declared and undeclared, does not depend on the time span: it simply depends on the model parameters (the discount rate, the expected returns of the two assets, the tax related parameters).

### 3.1 Optimal tax evasion

The optimal level of tax evasion (3) depends on investor preferences, on the variables of the tax system ( $\tau$ ,  $\tau_G$ ,  $\lambda$ , and  $\theta$ ), and on the return of the riskless asset  $r$ . Neither the return nor the volatility of the risky asset (i.e.  $\mu$  and  $\sigma$ ) affect evasion.

The decision to evade is negatively correlated to risk aversion. An infinitely risk averse individual ( $\delta = \infty$ ) will invest only in the riskless asset and by definition he will not be able to evade.

Our model shows that tax evasion can be reduced by using several instruments and in some circumstances it may also disappear. These findings can be summarised as follows.

1. The amount of evasion negatively depends on  $\tau_G$ . When  $\tau_G$  increases then: (i) evasion decreases, (ii) the investment in the riskless asset decreases, (iii) and the investment in the risky asset increases. In fact, we have

$$\frac{\partial \frac{w_0^*(t)S_0(t)}{R(t)}}{\partial \tau_G} = -\frac{1}{\theta} \frac{1}{\delta} \left( \frac{(1-\tau_G)r\tau}{(1-\tau)\theta\lambda} \right)^{-\frac{1}{\delta}-1} \frac{r\tau}{(1-\tau)\theta\lambda} < 0.$$

2. The amount of evasion is positively correlated with  $\tau$ , in fact

$$\frac{\partial \frac{w_0^*(t)S_0(t)}{R(t)}}{\partial \tau} = \frac{1}{\delta} \left( \frac{(1-\tau)\theta\lambda}{(1-\tau_G)r\tau} \right)^{\frac{1}{\delta}-1} \frac{\lambda}{(1-\tau_G)r\tau} \frac{1}{\tau} > 0,$$

i.e. the tax rate on the risky asset increases tax evasion. Our result is in line with Lyn (2001) and with the most recent empirical literature (Cebula and Feige, 2011). Ytzaky (1974) counterintuitive result that tax evasion reduces if the rate is increased does not seem to be confirmed for our model. This may not be surprising since we use Allingham and Sandmo (1972) approach to make the fine proportional to the amount of capital evaded rather than to the tax evaded. It is however very important to note the countervailing effect of the tax rate. The tax rate on the risky

assets is positively correlated with tax evasion while the tax rate on the riskless-evasion free asset reduces tax evasion.

**Corollary 5** *The elasticity of optimal tax evasion (in absolute value) is higher with respect to  $\theta$  than with respect to  $\lambda$ .*

**Proof.** The elasticity of  $\frac{w_0^*(t)S_0(t)}{R(t)}$  with respect to  $\theta$  is

$$\frac{\partial \frac{w_0^*(t)S_0(t)}{R(t)}}{\partial \theta} \frac{\theta}{\frac{w_0^*(t)S_0(t)}{R(t)}} = -1 - \frac{1}{\delta} \frac{\left(\frac{(1-\tau)\theta\lambda}{(1-\tau_G)r\tau}\right)^{\frac{1}{\delta}}}{1 - \left(\frac{(1-\tau)\theta\lambda}{(1-\tau_G)r\tau}\right)^{\frac{1}{\delta}}} < 0,$$

and the elasticity with respect to  $\lambda$  is

$$\frac{\partial \frac{w_0^*(t)S_0(t)}{R(t)}}{\partial \lambda} \frac{\lambda}{\frac{w_0^*(t)S_0(t)}{R(t)}} = -\frac{1}{\delta} \frac{\left(\frac{(1-\tau)\theta\lambda}{(1-\tau_G)r\tau}\right)^{\frac{1}{\delta}}}{1 - \left(\frac{(1-\tau)\theta\lambda}{(1-\tau_G)r\tau}\right)^{\frac{1}{\delta}}} < 0.$$

It is obvious that in absolute value

$$\left| \frac{\partial \frac{w_0^*(t)S_0(t)}{R(t)}}{\partial \theta} \frac{\theta}{\frac{w_0^*(t)S_0(t)}{R(t)}} \right| > \left| \frac{\partial \frac{w_0^*(t)S_0(t)}{R(t)}}{\partial \lambda} \frac{\lambda}{\frac{w_0^*(t)S_0(t)}{R(t)}} \right|.$$

■

From a policy point of view this implies that in order to fight evasion in a more effective way, the Government should increase the fine  $\theta$  rather than increase the number of controls.

This result is in line with the recent empirical evidence (Cebula and Feige, 2011) which shows that tax evasion is decreasing in the audit rate, but it also may explain why the number of audits are decreasing through time (Slemrod, 2007). If fines are more effective in reducing tax evasion and less costly than controls, it may make sense to reduce the latter. On the other hand, fines should be credible: when they are quite high the social cost may be too high to be enforced and for this reason controls are still necessary.

## 4 Government revenue

In this section we study the impact of tax evasion on Government budget. The general idea is that evasion reduces the Government's revenue and forces "honest" taxpayers to bear an unfair burden of the cost of public activities. This is certainly true for the amount of tax evasion that goes undetected, but to evaluate the impact on Government budget we need to take account of the (net) revenue that can be derived from tax audit.

If we call  $\Theta(t)$  the total Government revenue, then in differential term we have

$$d\Theta(t) = \tau_G w_G(t) dG(t) + \tau w(t) dS(t) + w_0(t) S_0(t) \theta d\Pi(t),$$

whose expected value is

$$\mathbb{E}_t[d\Theta(t)] = (\tau_G w_G(t) G(t) r + \tau w(t) S(t) \mu + w_0(t) S_0(t) \theta \lambda) dt.$$

Investor's wealth is

$$R(t) = w_G(t) G(t) + w(t) S(t) + w_0(t) S_0(t),$$

hence the expected revenue from capital income tax will be equal to (where we have substituted for  $w_G(t) G(t)$ ):

$$\mathbb{E}_t[d\Theta(t)] = R(t) \left( \tau_G r + \frac{w(t) S(t)}{R(t)} (\tau \mu - \tau_G r) + \frac{w_0(t) S_0(t)}{R(t)} (\theta \lambda - \tau_G r) \right) dt.$$

If we substitute the optimal values of both  $\frac{w(t)S(t)}{R(t)}$  and  $\frac{w_0(t)S_0(t)}{R(t)}$  we then obtain

$$\mathbb{E}_t[d\Theta(t)] = R(t) \left( \begin{array}{l} \tau_G r + \frac{1}{\delta} \frac{(1-\tau)\mu - (1-\tau_G)r}{(1-\tau)^2 \sigma^2} (\tau \mu - \tau_G r) \\ + \frac{1}{\theta} \left( 1 - \left( \frac{(1-\tau)\theta\lambda}{(1-\tau_G)r\tau} \right)^{\frac{1}{\delta}} \right) \left( \theta \lambda - \tau \frac{\mu - \tau_G r}{1-\tau} \right) \end{array} \right) dt. \quad (10)$$

Government revenue depends on all of the market and fiscal variables, as one might expect. The first two terms of the equation represent the expected revenue in the absence of tax evasion. Let us now concentrate on the third term which depends also on the tax audit parameters  $\lambda$  and  $\theta$ :

$$F(\theta, \lambda) \equiv \frac{1}{\theta} \left( 1 - \left( \frac{(1-\tau)\theta\lambda}{(1-\tau_G)r\tau} \right)^{\frac{1}{\delta}} \right) \left( \theta \lambda - \tau \frac{\mu - \tau_G r}{1-\tau} \right).$$

If evasion is convenient (i.e. if condition (6) holds) then it is easy to show that this function is always negative.

**Proposition 6** *The Government's expected revenue (10) reduces when assets are evaded, i.e.*

$$\mathbb{E}_t[d\Theta(t)] < \mathbb{E}_t \left[ d\Theta(t) \Big|_{\frac{w_0(t)S_0(t)}{R(t)}=0} \right].$$

**Proof.** Let us assume that condition (6) holds. Then

$$\frac{1}{\theta} \left( 1 - \left( \frac{(1-\tau)\theta\lambda}{(1-\tau_G)r\tau} \right)^{\frac{1}{\delta}} \right) \left( \theta \lambda - \tau \frac{\mu - \tau_G r}{1-\tau} \right) < 0,$$

implies

$$\theta\lambda - \tau \frac{\mu - \tau_G r}{1 - \tau} < 0,$$

which can be written as

$$1 - \frac{\tau(\mu - r) + (1 - \tau_G)\tau r}{(1 - \tau)\theta\lambda} < 0,$$

or

$$1 - \frac{(1 - \tau_G)\tau r}{(1 - \tau)\theta\lambda} - \frac{\tau}{1 - \tau} \frac{\mu - r}{\theta\lambda} < 0.$$

Now, since the sum of the two first terms is negative because of condition (6), and the last term is negative because  $\mu > r$ , then the inequality holds. ■

## 5 Policy implications

Without tax evasion, an optimal portfolio is highly liquid only for high value of investor's risk aversion. Nevertheless, when evasion is possible, a high liquidity is a by-product of evasion and the optimal asset allocation cannot be used any longer for measuring the investors' risk aversion. Instead, in our framework, a high liquidity can be used for targeting audits.

Tax evasion has an interesting countervailing effect on the distortion created by a symmetric tax system. In fact, through tax evasion Government shares the expected losses with investors only for the assets that have been declared. This increases the risk borne by the representative investor and cause a re-allocation among financial assets. The allocation with tax evasion will then closer to the one we would expect without taxation.

A first interesting trade-off emerges in this context: proposition 1 shows that the tax system causes a distortion in the optimal portfolio allocation due to the risk sharing mechanism determined by the tax rebates. To reduce such distortion the tax rate for riskless assets should be lower than that for the risky one; however we have also shown that in order to reduce evasion the tax rate on the riskless asset should be increased.

Our model considers a representative individual and it does not allow to draw policy implications concerning equality and fair distribution of the tax burden. Nevertheless, some equality issues arise. The tax rate on riskless assets should be increased to counterbalance tax evasion. In this way it is possible to reduce the level of tax evasion to tax evaders indirectly using a higher rate on what they must declare. However, if consumers are heterogeneous, also risk averse individuals have a portfolio biased towards riskless assets. These individuals are also less prone to tax evasion and yet they will be taxed at the same rate as evaders. If there a correlation between risk aversion and income, the risk is to get a regressive tax system.

## 6 Conclusions and directions for future research

The effects of taxation on household portfolio has long been debated in the literature. Theoretical models predict that under different taxation systems, the optimal portfolio allocation depends not only on the risk/return profile, but also on the tax characteristics concerning rate and timing (Poterba 2002 and Sule, 2010). Surprisingly, tax evasion has not received the same attention in spite of its policy implications.

The model proposed in this paper aims at bridging this gap by examining the intertemporal portfolio problem for an investor with the opportunity investing both in a taxable, risk free asset that cannot be evaded and a risky asset whose income can be evaded.

The framework we use is symmetric and very simple, yet the results are surprisingly rich. From a theoretical point of view our model contributes to explain the observed composition of individual portfolio, usually biased towards liquidity, and from a policy point of view it address some important questions as concerns the best instruments to reduce tax evasion.

## A Optimization

If we call  $J(t, R)$  the value function, then the Hamilton-Jacobi-Bellman (HJB) equation is

$$0 = \frac{\partial J(t, R)}{\partial t} - \rho J(t, R) + \max_{w(t), w_0(t), c(t)} \left[ \begin{aligned} & \frac{c(t)^{1-\delta}}{1-\delta} + \frac{\partial J(t, R)}{\partial R} R(t) (1 - \tau_G) r \\ & + \frac{\partial J(t, R)}{\partial R} w(t) S(t) ((1 - \tau) \mu - (1 - \tau_G) r) \\ & + \frac{\partial J(t, R)}{\partial R} (w_0(t) S_0(t) (\mu - (1 - \tau_G) r) - c(t)) \\ & + \frac{1}{2} \frac{\partial^2 J(t, R)}{\partial R^2} (w(t) (1 - \tau) S(t) + w_0(t) S_0(t))^2 \sigma^2 \\ & + (J(t, R - w_0(t) S_0(t) \theta) - J(t, R)) \lambda \end{aligned} \right],$$

whose boundary (final) condition is

$$J(T, R) = \chi \frac{R(T)^{1-\delta}}{1-\delta}.$$

The first order conditions on consumption, declared asset, and undeclared asset are

$$\begin{aligned} c^*(t) &= \left( \frac{\partial J(t, R)}{\partial R} \right)^{-\frac{1}{\delta}}, \\ w^*(t) (1 - \tau) S(t) + w_0^*(t) S_0(t) &= - \frac{\frac{\partial J(t, R)}{\partial R} (1 - \tau) \mu - (1 - \tau_G) r}{\frac{\partial^2 J(t, R)}{\partial R^2} (1 - \tau) \sigma^2}, \\ w^*(t) (1 - \tau) S(t) + w_0^*(t) S_0(t) &= - \frac{\frac{\partial J(t, R)}{\partial R} \mu - (1 - \tau_G) r}{\frac{\partial^2 J(t, R)}{\partial R^2} \sigma^2} \\ &\quad + \frac{1}{\frac{\partial^2 J(t, R)}{\partial R^2}} \frac{\partial J(t, R - w_0^*(t) S_0(t) \theta)}{\partial (R - w_0^*(t) S_0(t) \theta)} \frac{\theta \lambda}{\sigma^2}. \end{aligned}$$

Please note that if either  $\lambda = 0$  or  $\theta = 0$  (i.e. there are either no jumps or evasion is never punished), the two last conditions are not compatible. In fact, they become

$$\begin{aligned} w^*(t) (1 - \tau) S(t) + w_0^*(t) S_0(t) &= - \frac{\frac{\partial J(t, R)}{\partial R} (1 - \tau) \mu - (1 - \tau_G) r}{\frac{\partial^2 J(t, R)}{\partial R^2} (1 - \tau) \sigma^2}, \\ w^*(t) (1 - \tau) S(t) + w_0^*(t) S_0(t) &= - \frac{\frac{\partial J(t, R)}{\partial R} \mu - (1 - \tau_G) r}{\frac{\partial^2 J(t, R)}{\partial R^2} \sigma^2}, \end{aligned}$$

and the optimization problem does not have any feasible solution (in this case, there exists a solution if and only if  $\tau = 0$ ).

If we equate the left hand sides of the second and third equation we obtain that  $w_0^*(t)$  must solve

$$\frac{\partial J(t, R)}{\partial R} \frac{(1 - \tau_G) r \tau}{(1 - \tau) \theta \lambda} = \frac{\partial J(t, R - w_0^*(t) S_0(t) \theta)}{\partial (R - w_0^*(t) S_0(t) \theta)}.$$



Thus, we can compute the values of  $w_0^*(t)$ ,  $w^*(t)$ , and  $c^*(t)$  as functions of  $J(t, R)$  which must solve the HJB differential equation. One of the most common method to find  $J(t, \hat{R})$  is to try a guess function. Here, we use

$$J(t, R) = F(t)^\delta \frac{R(t)^{1-\delta}}{1-\delta},$$

where  $F(t)$  must be found in order to solve the HJB differential equation, with the boundary condition

$$F(T) = \chi^{\frac{1}{\delta}}.$$

Accordingly, the optimal values of the decision variables are

$$\begin{aligned} c^*(t) &= \frac{R(t)}{F(t)}, \\ \frac{w_0^*(t) S_0(t)}{R(t)} &= \frac{1}{\theta} \left( 1 - \left( \frac{(1-\tau)\theta\lambda}{(1-\tau_G)r\tau} \right)^{\frac{1}{\delta}} \right), \\ \frac{w^*(t) S(t)}{R(t)} &= \frac{1}{\delta} \frac{(1-\tau)\mu - (1-\tau_G)r}{(1-\tau)^2 \sigma^2} - \frac{1}{1-\tau} \frac{w_0^*(t) S_0(t)}{R(t)}, \end{aligned}$$

and the value of the function  $F(t)$  must solve

$$0 = \frac{\partial F(t)}{\partial t} - \phi F(t) + 1,$$

where

$$\begin{aligned} \phi &\equiv \frac{\rho + \lambda}{\delta} - \frac{1-\delta}{\delta} (1-\tau_G)r \left( 1 + \frac{\tau}{(1-\tau)\theta} \right) \\ &\quad - \frac{1}{2} \frac{1-\delta}{\delta^2} \left( \frac{(1-\tau)\mu - (1-\tau_G)r}{(1-\tau)\sigma} \right)^2 - \lambda \left( \frac{(1-\tau)\theta\lambda}{(1-\tau_G)r\tau} \right)^{\frac{1-\delta}{\delta}}. \end{aligned}$$

Given the boundary condition, the unique solution of this equation is

$$F(t) = \frac{1}{\phi} + \left( \chi^{\frac{1}{\delta}} - \frac{1}{\phi} \right) e^{-\phi(T-t)}.$$

## B References

### References

- [1] Allingham, Michael G., and Agnar Sandmo, 1972. Income Tax Evasion: A Theoretical Analysis, *Journal of Public Economics*, 1: 323-338
- [2] Bergstresser, D. and J. Poterba. "Asset Allocation And Asset Location: Household Evidence From The Survey Of Consumer Finances," *Journal of Public Economics*, 2004, v88(9-10, Aug), 1893-1915.
- [3] Buehn A., F. Schneider (2009) *Economics: The Open-Access, Open-Assessment E-Journal*, Vol. 1, 2007-9 (Version 2). <http://www.economics-ejournal.org/economics/journalarticles/2007-9>
- [4] Caplin, A. and J. Leahy (2001), "Psychological Expected Utility Theory and Anticipatory Feelings", *Quarterly Journal of Economics*, 116(1):55-80
- [5] Canner, Niko & Mankiw, N Gregory & Weil, David N, 1997 "An Asset Allocation Puzzle," *American Economic Review*, American Economic Association, vol. 87(1), pages 181-91, March.
- [6] Cebula R. and E.L. Feige (2011) America's Underground Economy: Measuring the Size, Growth and Determinants of Income Tax, MPRA Paper No. 29672, march, available at <http://mpra.ub.uni-muenchen.de/29672/>
- [7] Dalamagas B, 2011, A Dynamic Approach to Tax Evasion, *Public Finance Review*, 39:309-326
- [8] Davidson C., L. Martin and J.D. Wilson, 2007, Efficient black markets, *Journal of Public Economics*, 91(7-8), 1575-1590
- [9] Dzhumashev, R Gahramanov, E, 2011, Comment on "A dynamic portfolio choice model of tax evasion: Comparative statics of tax rates and its implication for economic growth", *Journal of Economics Dynamics and Control*, 35, 253-256
- [10] Engel, Eduardo M. R. A. and Hines Jr., James R., Understanding Tax Evasion Dynamics (January 1999). NBER Working Paper No. W6903. Available at SSRN: <http://ssrn.com/abstract=147433>
- [11] Landberg, N. (2008) Optimal financing for growth firms, *Journal of Financial Intermediation*, 17, 379-406
- [12] Fuest C., N. Riedel (2009) Tax evasion, tax avoidance and tax expenditures in developing countries: A review of the literature, Report prepared for the UK Department for International Development (DFID), available at <http://www.sbs.ox.ac.uk/centres/tax/Documents/reports/TaxEvasionReportDFIDFINAL1906.pdf>

- [13] Levaggi R., 2007, Tax Evasion and the Cost of Public Sector Activities, *Public Finance Review*, **35**, 572-585
- [14] Lin WZ, Yang CC, 2001, A dynamic portfolio choice model of tax evasion: Comparative statics of tax rates and its implication for economic growth", *Journal of Economics Dynamics and Control*, 25, 1827-1840
- [15] Cowell, Frank, 1985. The Economic Analysis of Tax Evasion. *Bulletin of Economic Research*, 37(3): 163-194.
- [16] Poterba, J.M (2002) Taxation, Risk-Taking, and Portfolio Behavior, in A. Auerbach and M. Feldstein, eds., *Handbook of Public Economics: Volume 3*, Amsterdam: North Holland, 2002, pp. 1109-71.
- [17] Richter, Wolfram, and Robin W. Boadway, 2005. Trading off tax distortion and tax evasion. *Journal of Public Economic Theory*, 7(3):361-381
- [18] Rosen, Harvey S., 2005. *Public Finance*. McGraw-Hill/Irwin, New York.
- [19] Sandmo, Agnar, 2005. The theory of tax evasion: a retrospective view. *National Tax Journal*, LVII(4): 643-663
- [20] Schneider, Friedrich, 2003. The Development of the Shadow Economies and Shadow Labor Force of 22 Transition and 21 OECD Countries. IZA D.P. n.514 available at: <http://www.iza.org>
- [21] Schneider Friedrich, 2005. Shadow economies around the world: what do we really know? *European Journal of Political Economy*. 21, 598-642
- [22] Slemrod, Joel. 2007. "Cheating Ourselves: The Economics of Tax Evasion." *Journal of Economic Perspectives*, 21(1): 25-48.
- [23] Slemrod, Joel, and Shlomo. Yitzhaki, 2002. Tax Avoidance, Evasion, and Administration. *Handbooks in Economics*, vol. 4. Amsterdam; London and New York: Elsevier Science, North-Holland
- [24] Sule, A, Atalay K, Crossley T Jeon S.(2010) New evidence on taxes and portfolio choice, *Journal of Public Economics*, 94,813-823
- [25] Yitzhaki, Shlomo, 1974. A Note on Income Tax Evasion: A Theoretical Analysis. *Journal of Public Economics* 3, 201-202.
- [26] Yitzhaki, Shlomo, 1979. A Note on Optimal Taxation and Administrative Costs. *American Economic Review*, 69(2):475-480.