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POPULATION IN FACTOR ACCUMULATION-BASED GROWTH

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Population in Factor Accumulation-based Growth

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Abstract

This paper analyzes the conditions under which, within a two-sector endogenous growth model with human and physical capital accumulation but without R&D-driven disembodied technological progress, it is possible to observe an ambiguous effect of population growth on economic growth, as empirical evidence suggests. We present three models. In each of them the engine of long-run growth is human capital accumulation. Population growth exerts ambiguous effects on economic growth only when human and physical capital are complementary for each other in the production of new human capital. This result is explained in terms of the interplay between the "dilution" and "accumulation" effects. In accordance with the growth literature exhibiting endogenous human capital accumulation and R&D activity, we also find that income growth can be positive even with stable population, that both the growth rate and the level of per-capita income are independent of population size, and finally that the level of per-capita income is proportional to per-capita human capital. We conclude that it is possible to reach the same results even without explicitly assuming endogenous and purposeful investment in research by firms.

***Key Words*:** Population Size and Growth; Scale Effects; Per-Capita Income; Economic Growth; Human Capital Investment; Physical Capital Accumulation

***JEL Codes*:** O41; J24; J10

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1. Introduction

One of the most evident results of the demographic transition that occurred or is still occurring in many countries is the increase of world population. From 1950 to 2000, this has more than doubled and, even more importantly, is forecast to reach 9.3 billion people by the year 2050 (United Nations, 2001). These numbers, together with the fact that a complete agreement on this issue has not emerged yet, both empirically and theoretically (see Ehrlich and Lui, 1997 for a survey), clearly suggest the importance of analyzing in depth the long-run effects of population growth on real per-capita income growth.

From a theoretical standpoint, proponents of the view that a larger population is detrimental to economic growth (see, among others, Solow, 1956 and Barro and Becker, 1988; 1989) base their argument on the belief that when a population size increases this leads to a dilution of reproducible resources. On the other hand, proponents of the optimistic view (population growth is beneficial to economic growth) insist on the positive effect that a larger population can exert on the rate of technological progress (Kuznets, 1967; Simon, 1981; Lee, 1988; Boserup, 1989; Kremer, 1993). In the presence of a larger population, the advocates of this idea claim, the likelihood of having a higher number of scientists and engineers is bigger and so is also a country's capacity of generating new scientific discoveries and, through this channel, of having higher per-capita income (*"More people means more Isaac Newtons and therefore more ideas"*, Jones, 2003). Given the *non-rival* nature of technical progress (the cost of inventing a new idea is independent of the number of individuals who use it), however, all growth models sharing the optimistic view that population spurs economic growth via technical progress display some scale effect. This means that in such models there is a positive influence either of population size or of population growth on per-capita income growth.

The huge body of empirical work conducted in recent times by economists and demographers rejects the hypothesis that population size might affect positively economic growth (*strong* scale effect) and in general seems to favor the conclusion that there can be an ambiguous correlation between population and economic growth rates. This emerges clearly from two recent contributions by Kelley and Schmidt (2003) and Bloom *et al*. (2003), respectively. While Kelley and Schmidt (2003) start their review on economic and demographic change by writing: *"No empirical finding has been more important to conditioning the "population debate" than the widely-obtained statistical result showing a general lack of correlation between the growth rates of population and per-capita output"*, according to Bloom *et al*. (2003, p.17): *"…Though countries with rapidly growing populations tend to have more slowly growing economies…, this negative correlation typically disappears (or even reverses direction) once other factors …are taken into account. …In other words, when controlling for other factors, there is little cross-country evidence that population growth impedes or promotes economic growth. This result seems to justify a third view: population neutralism"*.[1]

One possible account for this ambiguous effects of population growth is that the economic growth consequences of a new birth are not stable over time, since they are likely to be initially negative (because of

[1] Kelley (1988, p. 1686) was among the first to conclude that, depending on the country, population change may contribute, deter or even have no impact on economic development.

the cost of child-rearing), then positive during the working years (because of the larger labor force it generates), and then again negative during retirement (because of the cost of expenditures on elders). Hence, and depending on the relative size of each of these effects at a given point in time for a certain society, population change may have either a positive, or a negative, or else a negligible influence on the rate of growth of per-capita income across countries (see Crenshaw *et al*., 1997).

Our contribution proposes an alternative explanation of this ambiguous result based on the fact that population growth affects economic growth not only directly (*dilution effect*) but, eventually, also indirectly through factor accumulation. We especially focus on human and physical capital as reproducible inputs for two reasons. The first is that, while recognizing that human capital accumulation is one of the most powerful engines of growth of a nation (Lucas, 1988, 1993), there has recently been a rebirth of confidence *"…that explicit neoclassical growth models in the style of Solow (1956) can be adapted to fit the observed behavior of rich and poor economies alike"* (Lucas, 1993, p.253). Together, these two claims explain our interest in extending the original Solow's work through the addition of skill acquisition decisions by agents in order to have a theoretical model in which growth is sustained by human capital investment and real per-capita income is proportional to the stock of per-capita human capital. The second reason has to do with the fact that in analyzing the long-run relationship between population (size and growth) and per-capita income (level and growth) theoretical growth models consider for the most part economic environments where either solely technological change or both technological change *and* human capital accumulation are endogenous. In other words, in this literature physical capital accumulation plays for the most part a negligible role. According to Jones (1999), the first class of models (those with endogenous technological change and *no* investment in human capital) can be divided into three groups. The first group include papers such as those of Romer (1990), Grossman and Helpman (1991) and Aghion and Howitt (1992) that generate a positive relationship between per-capita income growth and population *size*. As already mentioned, this result (*strong* scale effect) is generally rejected on empirical grounds. The second group, instead, includes the so-called *semi-endogenous* growth models (e.g., Jones, 1995; Kortum, 1997 and Segerstrom, 1998). Such papers suggest not only that economic growth depends positively on population *growth* (as opposed to population *size*), but also that growth in real per-capita income is zero in the absence of any population change. We have said before that available evidence does not support the prediction that income growth is unambiguously and positively correlated with population growth.[2] Finally, the third group includes models (*e.g.*, Young, 1998; Peretto, 1998, Dinopoulos and Thompson, 1998 and Howitt, 1999) that are able to explain why we can observe positive growth in per-capita incomes even without population growth. On the other hand, the second class of theoretical models dealing with the long-run connections between population and economic growth (those with endogenous technological change *and* human capital accumulation), includes papers such as Dalgaard and Kreiner (2001) and Strulik (2005), among others. The main conclusions of this branch of the literature are that: 1) population growth is neither necessary nor conductive for economic growth; the long-run level of per-capita income is proportional to the skill level of the average individual in the population and

[2] See also Strulik (2005) and Laincz and Peretto (2006).

(for given level of per-capita skills) independent of population size (Dalgaard and Kreiner, 2001); 2) the long-run growth rate of the economy is independent of population size and is compatible with a stable population; long-run per-capita income growth is ambiguously correlated with population growth in the sense that the relationship between these two growth rates can be either positive, or negative or else equal to zero (Strulik, 2005).

Our contribution has two main objectives. The first is to study the conditions on the structure of the model under which, even in an economy that accumulates just human and physical capital and where there is no (disembodied) technological progress driven by purposeful R&D activity by firms, we can observe an ambiguous effect of population growth on economic growth. The second, instead, consists in verifying that the main, and in most cases empirically supported, theoretical results found by the works reviewed above - and dealing either solely with endogenous technological change or with both endogenous technological change *and* human capital accumulation- may hold in this economy as well (namely, the absence of *strong* scale effects; the independence of the level of per-capita income of population size; the existence of a positive relation between per-capita income and per-capita human capital and, finally, the possibility of having positive per-capita income growth even without any population change).

In order to meet these objectives, in the following sections we present three different models. In the first two we shall assume, respectively, exogenous and endogenous saving rate. We find that population growth exerts a negative effect on economic growth in both models (this is the traditional *dilution effect*). However, when individuals choose endogenously how much to save (model II) population growth can also have a neutral influence on economic growth. This result corresponds to the empirical finding of a weak, even though frequently negative correlation between growth rates of population and income per capita (Bloom *et al.*, 2003). In the third model we further extend our analysis to the case where physical and human capital can interact with each other in the production of new human capital. We find that when the two types of capital are substitutes for each other in the education sector, the effect of population growth on per-capita income growth is always negative. Instead, if human and physical capital are complementary for each other, the impact of population change on real per-capita income growth becomes ambiguous. The intuition is simple. For given per-capita physical capital stock, an increase of population causes the aggregate physical capital to rise. If physical and human capital are substitutes (in the sense that the larger amount of physical capital being now available in the economy deters the demand and, thus, the consequent supply of human capital), the increase of population size, together with the reduction of the aggregate human capital stock, determines an unambiguous decline of the per-capita level of skills and, via this channel, a lower per-capita income growth rate. On the other hand, if physical and human capital are complements (the increase in the supply of physical capital spurs the demand and, therefore, the consequent production of new human capital), the final effect on the per-capita level of skills and, hence, on per-capita income growth of an increase in population may be either positive, or negative, or else equal to zero.

In the three models we present in this paper the engine of long-run growth is human capital accumulation. Moreover, they also allow us concluding that: long-run per-capita income growth can be positive even without any population change; in equilibrium both the growth rate and the level of per-capita income are independent of population size; the long-run level of per-capita income is proportional to per-capita human capital.

The article is organized as follows: in Sections 2, 3 and 4 we present, respectively, our three models and discuss one at a time the main results we obtain from them. Section 5 summarizes and concludes.

2. Population and factor accumulation-based growth with exogenous saving rate: Model I

In an influential article Mankiw *et al*. (1992) have already and extensively analyzed the impact of human capital within an otherwise standard *Solovian* economy. Our paper introduces two major differences with respect to their theoretical model. First of all, we postulate that physical and human capital are produced by different technologies and that, as in Uzawa (1965) and Lucas (1988), the sector producing education (new human capital) is relatively intensive in human capital as an input.[3] Secondly, we make the distribution of human capital across alternative uses (production of goods and education) endogenous.[4] More formally we assume that in this economy there are two perfectly-competitive sectors. The final output sector produces homogeneous consumption goods by combining physical and human capital through a constant returns to scale technology. As in the Solow's model (1956), physical capital is accumulated by saving in each period a positive, constant and exogenous fraction of total output. Following Barro and Sala-i-Martin (2004, Chap.5, p. 240) the aggregate stock of human capital (H_t) is given by the number of workers (L_t) multiplied by the average skill level of each worker (h_t). Thus, it may rise over time not only because of increases in the population size, but also because of a rising quality of each individual. Human capital is employed in two alternative activities. The fraction u_t of it is used to produce consumption goods and the fraction (1-u_t), instead, is used to produce new human capital. Population (L_t) grows at a constant, exogenous rate (n) and its size at initial time is normalized to one ($L_0 \equiv 1$).

2.1 Production

In order to produce final goods (Y) human (H_Y) and physical (K) capital are combined through a constant returns to scale Cobb-Douglas technology:

$$Y_t = AK_t^{\alpha}\left(H_{Yt}\right)^{1-\alpha}, \qquad \alpha \in (0;1), \tag{1}$$

with A denoting total factor productivity and α and $(1-\alpha)$ being the shares of output going to K and H_Y, respectively. The hypothesis $\alpha \in (0;1)$ ensures that physical and human capital are both necessary in the

[3] In Mankiw *et al.*'s model, constant fractions of GDP are invested in physical and human capital accumulation. As it is shown in Cohen (1996), however, this formulation is not supported by data.

[4] In Mankiw *et al.* (1992), the allocation of the available resources (final consumption goods) between physical and human capital accumulation activities is exogenously given.

production of final output. We expressly omit any disembodied technological progress[5] from the model and assume that A is a positive constant. In particular, and without any loss of generality, we set $A=1$. Under this hypothesis, the technology for the production of consumption goods reads as:

$$Y_t = K_t^{\alpha}\left(H_{Yt}\right)^{1-\alpha}, \qquad \alpha \in (0;1). \tag{1'}$$

The production technology (1') and the assumption $A=1$ are used for two main reasons. First of all because in this paper we intend to show that most of the empirically-supported results on the long-run relationship between population and economic growth found by recent literature dealing with endogenous disembodied technical change can be recovered by analyzing an economy that accumulates two types of capital (human and physical capital), in which there is no disembodied technological progress and where human capital investment is the solely source of long-run growth in per capita incomes. Secondly, Dalgaard and Kreiner (2001) and Bucci (2008) have, among others, recently analyzed the effects of population (size and growth) on the level and the growth rate of per capita income in a model where human capital accumulation *and* disembodied technical progress (resulting from increasing specialization) are both endogenous. In their paper they use an aggregate technology for the production of final goods of the type:

$$Y_t = \left(\frac{H_{Yt}}{N_t}\right)^{1-\alpha} \int_0^{N_t} x_{jt}^{\alpha} dj ,$$

where $[0; N_t]$ - $N_t \in [0;\infty)$ - is the range of specialized intermediate inputs (x_{jt}) existing at time *t*. This production function exhibits constant returns to scale to *rival* (H_{Yt} and x_{it}) as well as *reproducible* (H_{Yt} and N_t) factors. If we define the aggregate capital stock existing at time *t* as $K_t \equiv \int_0^{N_t} x_{jt} dj$, implying that one unit of physical capital (forgone consumption) can be transformed instantaneously into one unit of any intermediate good for which a design has already been discovered, it is immediate to show that in a symmetric equilibrium in which $x_{jt} = x_t$ for each j, the aggregate production function written above can be recast as:

$$Y_t = K_t^{\alpha}\left(H_{Yt}\right)^{1-\alpha}.$$

Clearly, this technology coincides with (1') and, hence, can be seen as a reduced form of a more complicated aggregate production function employing, as inputs, human capital, ideas and (symmetric) varieties of intermediate goods.

Final output is produced competitively and used as numeraire. Population at time *t* (L_t) consists of educated workers and is fully employed. Each worker is endowed with a stock of human capital (h_t) given by:

$$h_t \equiv \frac{H_t}{L_t}.$$

[5] It is well known that with a Cobb-Douglas production function there is no need to distinguish among *Harrod-neutral* (labor-augmenting), *Solow-neutral* (capital-augmenting) and/or *Hicks-neutral* technical change.

From the definition of h_t it follows that the total stock of human capital available in the economy at *t* (H_t) can be written as:

$$H_t = h_t L_t \,. \tag{2}$$

In (2) h_t represents the human capital possessed by each member of the population (*i.e.*, per-capita human capital). In this economy all the available human capital (H_t) can be used in two alternative activities. A fraction u_t of it (equal to H_{Yt}) is used to produce homogeneous final consumption goods, while its complement to one ($1-u_t$) is employed to accumulate new human capital. Hence, the human capital stock devoted to production of consumption goods at *t* is $H_{Yt} = u_t H_t = u_t\left(h_t L_t\right)$. Using this fact, we can re-write the aggregate production function as:

$$Y_t = K_t^{\alpha}\left(u_t h_t L_t\right)^{1-\alpha} \,. \tag{1”}$$

From (1”) real per-capita income is finally given by:

$$y_t \equiv \frac{Y_t}{L_t} = k_t^{\alpha}(u_t h_t)^{1-\alpha} \equiv f(k_t;u_t;h_t;\alpha), \qquad k_t \equiv \frac{K_t}{L_t} \,. \tag{3}$$

2.2 The Laws of Physical and Human Capital Accumulation

As in the model developed by Solow (1956), we assume that physical capital is accumulated through devoting to this activity a positive, exogenous and constant fraction (s) of total output. In the next model we shall endogenize the saving rate. Population grows at a rate (n) which is exogenous and constant, as well. Finally, we follow Barro and Sala-I-Martin (2004, p. 247) in assuming that human and physical capital depreciate over time at the same constant instantaneous rate. Without any loss of generality and only for the sake of simplicity, this common depreciation rate is set equal to zero in our model. Under these assumptions, the law of motion of physical capital in per-capita terms can be stated as:

$$\dot{k}_t = s\cdot f(k_t;u_t;h_t;\alpha) - nk_t\,, \qquad k_0 > 0\,, \qquad \frac{\dot{L}_t}{L_t} \equiv n \geq 0\,, \qquad s \in \left(0;1\right). \tag{4}$$

The term $-nk_t$ in (4) represents the well-known *dilution effect* in per-capita physical capital accumulation: it is the cost of bringing the level of physical capital of the newcomers up to the average level of the existing population (*k*).

As for human capital investment, and differently from Mankiw *et al.* (1992) where this factor is produced with the same production technology used for final output, we postulate that a fraction (equal to $1-u_t$) of the aggregate human capital stock available at time *t* is employed to produce new human capital in the education sector. More specifically, we assume that in the time unit $\dot{H}_t$ units of new human capital are obtained with the following production function:

$$\dot{H}_t = \sigma\left(1-u_t\right)H_t = \sigma\left(1-u_t\right)\left(h_t L_t\right), \qquad \sigma > 0\,, \qquad 0 \leq u_t \leq 1\,, \qquad H_0 > 0\,, \tag{5}$$

where σ represents the productivity of human capital in the production of new human capital. Hence, our model explicitly incorporates the same technology of human capital formation used by Uzawa (1965) and Lucas (1988). It is evident from equation (5) that the production of human capital occurs at constant returns to scale. This assumption, which is shared by many other models, can be justified by referring either to the existence of external effects in education (such that the decreasing returns to this activity at the individual level are converted into constant returns at the aggregate one) or to the fact that the production of new human capital involves not only time spent on pure educational activity but also other production factors (in this case, human capital should be considered in a broad sense).

Equation (5) and the definition of *h* can be used to find the law of per-capita human capital accumulation:

$$\dot{h}_t = \sigma\left(1-u_t\right)h_t - nh_t \qquad (6)$$

The term $-nh_t$ in (6) describes the same *dilution effect* in human capital accumulation we have already commented for physical capital investment (see 4). Intuitively, if the rate of investment in human capital $(1-u_t)$ were equal to zero, then per-capita human capital stock would diminish over time because of population growth (at rate *n*). In other words, population growth tends, *ceteris paribus*, to reduce the quality level of the average individual in the population.

2.3 Balanced Growth Path (BGP) equilibrium

In this section we determine the equilibrium value of the three endogenous unknowns of the model, that is the ratio of human to physical capital, the distribution of human capital across sectors and the growth rate of per-capita income. However, before doing that, we start with a formal definition of BGP equilibrium.

Definition: ***BGP Equilibrium***

A BGP equilibrium is a long-run equilibrium in which the endogenous state variables (i.e., human and physical capital) grow at constant (possibly positive) exponential rates.

A direct implication of this definition is that along the BGP equilibrium the shares of human capital devoted, respectively, to production of consumption goods (*u*) and to formation of new human capital (1-*u*) are also constant (see 5 and 6). More specifically, we have:

$$\gamma_h \equiv \frac{\dot{h}_t}{h_t} = \sigma\left(1-u\right) - n = \gamma_H - n\,, \qquad \gamma_H \equiv \frac{\dot{H}_t}{H_t}. \qquad (6')$$

Moreover, from (3) and (4) we obtain:

$$\frac{\dot{k}_t}{k_t} = \frac{sk_t^{\alpha}\left(uh_t\right)^{1-\alpha}}{k_t} - n \equiv \gamma_k = \gamma_K - n \qquad \Rightarrow \qquad s \cdot u^{1-\alpha}\left(\frac{h_t}{k_t}\right)^{1-\alpha} = \gamma_K\,, \qquad (7)$$

where $\gamma_K \equiv \dot{K}_t / K_t$ is a constant to be determined.

With s constant and exogenously given, and with u and γ_K also constant but endogenous, two fundamental conclusions can be drawn from equation (7). Along the BGP equilibrium:

- h_t and k_t grow at the same constant exponential rate, that is:

$$\gamma \equiv \gamma_k = \gamma_h = \sigma(1-u) - n; \tag{7'}$$

- $$\left(\frac{uh_t}{k_t}\right) = \left(\frac{\gamma_K}{s}\right)^{\frac{1}{1-\alpha}} = \left(\frac{\gamma + n}{s}\right)^{\frac{1}{1-\alpha}} = \left[\frac{\sigma(1-u)}{s}\right]^{\frac{1}{1-\alpha}}. \tag{7''}$$

From equation (3), and using the fact that $\gamma \equiv \gamma_k = \gamma_h$ and that u is constant, one obtains:

$$\frac{\dot{y}_t}{y_t} \equiv \gamma_y = \gamma_k = \gamma_h \equiv \gamma = \sigma(1-u) - n \tag{8}$$

Equation (8) suggests that, under the hypotheses of this simple model, there exists a BGP equilibrium along which income, physical capital and human capital (all measured in per capita terms) grow at a common and constant rate, being a linear function of *u*.

If capital markets are competitive, rational agents will invest in the type of capital that delivers the higher return, so that in equilibrium the two rates of return will have to be equalized if both forms of investment are taking place, (as we are assuming, see equation 1). Because the return (in terms of goods) from possessing one unit of human capital coincides with the wage rate (*w*, *i.e.* with the productivity of human capital in the production of goods), while the return from possessing one unit of physical capital coincides with the real interest rate (*r*, *i.e.* with the productivity of physical capital in the production of goods), in the long run it must be true that the two marginal products of capital are to be equal:

$$\alpha\left(\frac{uH_t}{K_t}\right)^{1-\alpha} = (1-\alpha)\left(\frac{K_t}{uH_t}\right)^{\alpha} \tag{9}$$

Solving (9) in $\left(\frac{uH_t}{K_t}\right)$ yields:

$$\left(\frac{uH_t}{K_t}\right) = \left(\frac{uh_t}{k_t}\right) = \left(\frac{1-\alpha}{\alpha}\right) \tag{10}$$

This is the standard result (see Barro and Sala-I-Martin, 2004, p.60 and p.241 for a one-sector model with physical and human capital) according to which in equilibrium the ratio of physical (k_t) to human capital (uh_t) employed in the production of goods must be equal to the ratio of their respective distributive shares (respectively, α and 1-α).[6] Equation (10) gives us another expression for the ratio of uh_t to k_t. Equalizing this equation to (7") yields a closed form solution for *u* and (1–*u*). Along the BGP equilibrium the

[6] Using equation (10) into (9), it is possible to verify that in equilibrium the common rate of return on the two kinds of assets (physical and human capital) is: $\alpha^{\alpha}(1-\alpha)^{1-\alpha} > 0$. See Barro and Sala-I-Martin (2004, p. 242) with $A = 1$ and no depreciation of physical and human capital ($\delta = 0$).

share of human capital employed by each agent to produce consumption goods and skills is, thus, respectively equal to:

$$u = 1 - \left(\frac{s}{\sigma}\right)\left(\frac{1-\alpha}{\alpha}\right)^{1-\alpha} \quad (11)$$

$$(1-u) = \left(\frac{s}{\sigma}\right)\left(\frac{1-\alpha}{\alpha}\right)^{1-\alpha} \quad (12)$$

Given *u*, it follows from equation (7') that the common balanced-growth rate of this economy is:

$$\gamma_y = \gamma_k = \gamma_h \equiv \gamma = \sigma(1-u) - n = s\left(\frac{1-\alpha}{\alpha}\right)^{1-\alpha} - n \quad (13)$$

Finally, given *u*, it follows from equation (7") – or, alternatively, from equation (10) – that:

$$\left(\frac{h_t}{k_t}\right) = \frac{\sigma(1-\alpha)}{\alpha\left[\sigma - s\alpha^{\alpha-1}(1-\alpha)^{1-\alpha}\right]} \quad (14)$$

Along the BGP equilibrium *u*, γ and h_t / k_t are all constant and depend solely on the model's exogenous variables (*s* and *n*) and its technological (σ) and distributive (α) parameters. It is immediate to notice that, unlike the standard Solow model with exogenous saving rate, no (exogenous) technological progress and only one type of capital (*i.e.*, physical capital) accumulation, a permanent increase in *s* generates a permanent increase of the BGP equilibrium rate of per-capita income (γ).

PROPOSITION 1

The following relation among the model's exogenous parameters,

$$\sigma > \left(\frac{1-\alpha}{\alpha}\right)^{1-\alpha} > 1 > s\left(\frac{1-\alpha}{\alpha}\right)^{1-\alpha} > n \geq 0,$$

ensures simultaneously that:

- *the equilibrium growth rate of real per capita income (γ) is positive;*
- *the equilibrium ratio of human to physical capital (h_t / k_t) is also positive;*
- *the equilibrium distribution of human capital across sectors has an interior solution, i.e.* $u \in (0;1)$*;*
- *the restrictions on s, n and* σ *–i.e.,* $s \in (0;1)$, $n \geq 0$ *and* $\sigma > 0$*, see equations (4) and (5)– are checked.*

Proof

The proof of Proposition 1 follows immediately from equations (11), (13) and (14) and the restrictions on *s*, *n*, and σ stated in equations (4) and (5). ■

Notice that the constraint $\left(\frac{1-\alpha}{\alpha}\right)^{1-\alpha} > 1$ is satisfied for each $\alpha < 1/2$, which is in line with the evidence that the physical capital share in aggregate income (α) is, at least for the industrialized countries, approximately equal to 1/3 and in any case lower than 50%. Moreover, when α is set equal to 1/3, the inequality $s\left(\frac{1-\alpha}{\alpha}\right)^{1-\alpha} < 1$ is met for each $s < 0.63$ which, again, seem to be confirmed by data on the saving rate (the ratio of total savings to aggregate income) for industrialized countries.

After describing the model, we are now able to state the most relevant results.

THEOREM 1

In an economy that accumulates physical and human capital, in which the saving rate is exogenous and economic growth is driven solely by investment in human capital, real per-capita income growth (γ) depends positively on a constant (i.e., a positive combination of preference –s– and technological $-\alpha-$ exogenous parameters) and negatively on population growth. Given our restrictions on parameter values, economic growth is positive even in the absence of any change in population (n=0).

Proof

In order to prove the first part of the Proposition, see (13) and our assumptions on the parameter values. According to the same equation, $\frac{\partial \gamma_y}{\partial n} < 0$ and $\gamma_y = s\left(\frac{1-\alpha}{\alpha}\right)^{1-\alpha} > 0$ when $n = 0$. ■

THEOREM 2

In an economy that accumulates physical and human capital, in which the saving rate is exogenous and economic growth is driven solely by investment in human capital, the growth rate (γ) and the level (y_t) of real per-capita income are both independent of population size along the BGP equilibrium.

Proof

Equation (13) reveals that per-capita income growth (γ) depends on population growth (*n*), but not on population size (*L*).
From (3), along the BGP equilibrium per-capita real income can also be written as:

$$y_t \equiv \frac{Y_t}{L_t} = \left(\frac{h_t}{k_t}\right)^{-\alpha} u^{1-\alpha} h_t \,,$$

where *u* is the constant share of the available stock of human capital devoted to production of goods. Plugging (14) and (11) into the equation above yields:

$$y_t = \frac{1}{\sigma}\left(\frac{\alpha}{1-\alpha}\right)^{\alpha}\left[\sigma - s\left(\frac{1-\alpha}{\alpha}\right)^{1-\alpha}\right] h_0 e^{\gamma t} \,,$$

where $\left[\sigma - s\left(\frac{1-\alpha}{\alpha}\right)^{1-\alpha}\right] > 0$ and $\gamma \equiv \left[s\left(\frac{1-\alpha}{\alpha}\right)^{1-\alpha} - n\right] > 0$.

Simple inspection of the last equation suggests that, for given h_0 $(\equiv H_0 / L_0 = H_0)$, y_t is independent of population size, *L*, along the BGP equilibrium and is proportional to per-capita human capital. ■

In words, this simple model predicts that in an economy with two types of capital, in which economic growth is driven solely by human capital investment, and where there is no disembodied technological progress and where agents take their own saving rate as exogenously given, we should not observe any *strong* scale effect in the long run, since per-capita income growth is independent of population size along the BGP equilibrium (Theorem 2). Furthermore, and according to the same Theorem, the level of per-capita income is also independent of population size. This property allows the most populous countries of the world (for example, China and India) to be also among the poorest ones, as well as some of the smallest countries (*e.g.*, Hong Kong and Luxembourg) to be among the richest.[7] Finally, Theorem 1 states that population growth is neither necessary nor conducive to long-run growth in per-capita real income. When population growth deters economic growth this happens because of the presence of a *dilution effect* in the accumulation of physical and human capital (equations 4 and 6). In the next section we extend this model to the case of an endogenous saving rate.

3. Population and factor accumulation-based growth with endogenous saving rate: Model II

In this section we use the same hypotheses of model I, the only departure from that model being that now the saving rate is chosen optimally by agents. In particular, the technology for the production of goods is still given by equation (1'), implying that we continue to omit any disembodied technological progress from the analysis ($A=1$). Moreover, we still assume that the aggregate stocks of human and physical capital are not subject to any material obsolescence.

3.1 Consumers

Identical households live in a closed economy (there is no international trade in goods and/or services and no migrations across countries). They receive wages and interest income and purchase homogeneous final goods for consumption. Since households are identical to each other we can focus on the choices of a single dynastic family. The size of the representative household (L) rises over time at the constant and exogenous rate of population growth, $n \geq 0$. The household uses the income it does not consume to accumulate physical capital (K). Thus:

$$\dot{K}_t = Y_t - C_t, \qquad K_0 > 0$$

where $\dot{K}_t$ denotes (gross and net) physical capital investment, Y_t is aggregate output and C_t is aggregate consumption. Given the expression above, the law of motion of physical capital in per-capita terms ($k_t \equiv K_t / L_t$) reads as:

$$\dot{k}_t = (y_t - c_t) - nk_t = (r_t - n)k_t + w_t(u_t h_t) - c_t, \qquad y_t = r_t k_t + w_t(u_t h_t) \tag{15}$$

with $y_t\ (\equiv Y_t / L_t)$, $c_t (\equiv C_t / L_t)$ and $u_t h_t\ (\equiv H_{Yt} / L_t)$ denoting, respectively, per-capita income, per-capita consumption and the fraction of per-capita human capital employed to produce final goods. Households are

[7] Jones (2005), p. 1096.

competitive in that they take the real interest rate (r_t) and the wage rate (per unit of human capital services, w_t) as given. The fraction $(1-u_t)$ of the available stock of human capital is employed to produce new human capital. At the economy-wide level, the supply function of human capital (H_t) is still postulated to be:

$$\dot{H}_t = \sigma\left(1-u_t\right)H_t, \qquad \sigma > 0, \qquad 0 \le u_t \le 1, \qquad H_0 > 0. \qquad (16)$$

Given $\dot{H}_t$, the law of motion of human capital in per-capita terms (*h*) is given by:

$$\dot{h}_t = \sigma\left(1-u_t\right)h_t - nh_t. \qquad (16')$$

The terms $-nk_t$ and $-nh_t$ (respectively, in equations 15 and 16') continue to represent the *dilution effect* in the accumulation of per-capita physical and human capital.

With a constant intertemporal elasticity of substitution (*CIES*) instantaneous individual utility function, the objective of the representative household is:

$$\underset{\{c_t, u_t, k_t, h_t\}_{t=0}^{\infty}}{\text{Max}} \text{U}_0 \equiv \int_0^{\infty} \left(\frac{c_t^{1-\theta}-1}{1-\theta}\right) e^{-(\rho - \nu n)t} dt, \qquad \nu \in [0;1]; \qquad \rho > \nu n \ge 0; \qquad \theta > 1 \qquad (17)$$

s.t.:

$$\dot{k}_t = (r_t - n)k_t + w_t(u_t h_t) - c_t, \qquad n \ge 0 \qquad (15)$$

$$\dot{h}_t = \sigma\left(1-u_t\right)h_t - nh_t, \qquad (16')$$

$$\lim_{t\to\infty} \lambda_{kt} k_t = 0; \qquad \lim_{t\to\infty} \lambda_{ht} h_t = 0 \qquad (18)$$

$$k_0 > 0, \; h_0 > 0.$$

The household decides on the amount of per-capita consumption (c_t) and on the share of human capital to be devoted to production (u_t). Equations (17), (15) and (16') are, respectively, the household's overall discounted intertemporal utility, the per-capita budget constraint and the per-capita human capital accumulation function, whereas equation (18) gives the two transversality conditions. We denoted by $\rho > 0$ the pure rate of time preference and by $1/\theta$ the constant intertemporal elasticity of substitution in consumption. The hypothesis $\theta > 1$ is in line with most of the available evidence.[8] Following Strulik (2005, p.135), we introduced in the household's maximization problem a preference parameter (ν) controlling for the degree of *altruism* towards future generations. The case $\nu = 0$ defines the minimal degree of altruism, whereas the opposite case $\nu = 1$ defines the situation of *perfect altruism* (the household maximizes the utility of per capita consumption of all of its members, both actual and future, taking explicitly into account the fact that its own size grows over time). The assumption $\rho > \nu n$ ensures that *U* is bounded. The representative household's maximization stated above is a standard two-state-variables dynamic optimization problem, whose first order conditions are:

[8] See Hall (1988) and, more recently, Growiec (2006, pp. 17-19) for a concise survey.

(19) $\frac{e^{-(\rho-\nu n)t}}{c_t^{\theta}} = \lambda_{kt}$; (20) $\lambda_{kt} = \frac{\sigma}{w_t}\lambda_{ht}$;

(21) $\lambda_{kt}(r_t - n) = -\dot{\lambda}_{kt}$; (22) $\lambda_{kt}u_t w_t + \lambda_{ht}\left[\sigma(1-u_t) - n\right] = -\dot{\lambda}_{ht}$,

where λ_{ht} and λ_{kt} are the shadow prices associated with h_t and k_t, respectively. Equations (19) and (21) give the usual Ramsey-Keynes rule governing the optimal path of per-capita consumption. Equation (20) requires that the representative household accumulates human capital so that the value of an additional unit of human capital employed in manufacturing activities ($\lambda_{kt}w_t$) equals the value of an additional unit of human capital employed in education ($\sigma\lambda_{ht}$). Finally, equation (22) provides the evolution over time of the shadow price of human capital (λ_{ht}).

3.2 Balanced Growth Path (BGP) equilibrium

In order to compare the two models outlined above, we define the BGP equilibrium of this economy as follows:

Definition: ***BGP Equilibrium***

A BGP equilibrium is a long-run equilibrium in which the ratio of the two endogenous state variables and the shares of human capital devoted to production and education activities are both constant.

Thus, as in model I, we continue to be interested in analyzing the long-run predictions of a model in which u is constant and $\gamma \equiv \gamma_k = \gamma_h = \left[\sigma(1-u) - n\right]$ is constant (and possibly positive), as well -see equation 7'.

The following results hold along the BGP equilibrium (mathematical derivation of such results is in appendix A):

$$u = \frac{\sigma(\theta-1) + \rho - (\theta+\nu-1)n}{\sigma\theta} \quad (23)$$

$$1-u = \frac{(\sigma-\rho) + (\theta+\nu-1)n}{\sigma\theta} \quad (24)$$

$$\gamma_c = \gamma_k = \gamma_h = \gamma_y \equiv \gamma = \sigma(1-u) - n = \left(\frac{\sigma-\rho}{\theta}\right) - \left(\frac{1-\nu}{\theta}\right)n \quad (25)$$

$$\frac{h_t}{k_t} = \left(\frac{1-\alpha}{\sigma}\right)^{\frac{1}{\alpha}}\left[\frac{\sigma\theta}{\sigma(\theta-1) + \rho - (\theta+\nu-1)n}\right] \quad (26)$$

Equations (23) and (24) give the BGP equilibrium shares of human capital devoted, respectively, to production of goods and acquisition of new human capital. According to equation (25), per-capita consumption (*c*), physical capital (*k*), human capital (*h*) and income (y) grow in the long run at the same constant rate. Finally, equation (26) provides the equilibrium value for the ratio of per-capita human to physical capital (h_t / k_t).

PROPOSITION 2

With $\theta > 1$, $n \geq 0$ and $\nu \in [0;1]$, the following restrictions on the model's exogenous parameters,

$$\sigma > \rho + (1-\nu)n \geq \rho \geq \rho - (\theta + \nu - 1)n > \frac{-\rho + (\theta + \nu - 1)n}{(\theta - 1)}$$
$$\sigma > 0 \tag{27}$$
$$\rho\theta > \rho > (\nu + \theta - 1)n \geq \nu n \geq 0$$

guarantee simultaneously that:

- *the equilibrium growth rate of real per capita income (γ) is positive;*
- *the equilibrium ratio of human to physical capital (h_t / k_t) is also positive;*
- *the equilibrium distribution of human capital across sectors has an interior solution, i.e. $u \in (0;1)$;*
- *the two transversality conditions in equation (18) are checked;*
- *our assumption on σ in equation (16) is checked as well.*

Proof

See equations (23), (25) and (26) and appendix A. ■

In particular, notice that $\rho > (\nu + \theta - 1)n$ ensures that ρ is strictly positive, whereas the restriction $\sigma > \rho$, implied by (27), is a standard requirement in growth models with two state variables and endogenous human capital accumulation (see, as an example, Arnold, 1998, equation 1, p.85 and Strulik, 2005, equation 24, p. 135).

Theorems 3 and 4 summarize the most important results of model II. In what follows we assume that the restrictions on the parameters written in (27) are satisfied.

THEOREM 3

In an economy that accumulates physical and human capital, in which the saving rate is endogenous and economic growth is driven solely by investment in human capital, real per-capita income growth (γ) depends positively on a constant (i.e., a positive combination of preference – ρ and θ – and technological – σ – exogenous parameters). The effect of population growth on economic growth can be either negative or equal to zero (this happens when $\nu = 1$). Given our assumptions on parameter values, economic growth is positive even in the presence of a population growth rate equal to zero (n=0).

Proof

In order to prove the first part of the Proposition, see (25) and our assumptions on the parameter values (27). According to (25), $\frac{\partial \gamma_y}{\partial n} = -\left(\frac{1-\nu}{\theta}\right) \leq 0$. Finally, $\gamma_y = \frac{\sigma - \rho}{\theta} > 0$ when $n = 0$. ■

For given ρ and n, when the degree of *altruism* (ν) is low -meaning that the weight on the future size of the family is small- households are less patient, they discount future more and, hence, save less. In the long run this results in a lower common rate of investment in physical and human capital and, hence, in a lower growth rate of per-capita income. According to theorem 3, in the presence of egoistic agents ($\nu < 1$)

when the size of the dynastic family rises this increase would definitely exert a negative effect on γ. Under the assumption that most of growth models with optimizing individuals do with respect to ν (*i.e.* $\nu = 1$), equation 25 reveals immediately that, due to the structure of agents' preferences, real per-capita income growth is independent of population growth.

THEOREM 4

In an economy that accumulates physical and human capital, in which the saving rate is endogenous and economic growth is driven solely by investment in human capital, both the growth rate (γ) and the level (y_t) of real per-capita income are independent of population size along the BGP equilibrium.

Proof

Equation (25) reveals that per-capita income growth (γ) depends on population growth (*n*), but not on population size (*L*). From (3), along the BGP equilibrium per-capita real income can also be written as:

$$y_t \equiv \frac{Y_t}{L_t} = \left(\frac{h_t}{k_t}\right)^{-\alpha} u^{1-\alpha} h_t ,$$

where *u* is the constant share of the available stock of human capital devoted to production of goods. Plugging (23) and (26) into the equation above yields:

$$y_t = \left[\frac{\sigma(\theta-1)+\rho-(\theta+\nu-1)n}{(1-\alpha)\theta}\right] h_0 e^{\left[\left(\frac{\sigma-\rho}{\theta}\right)-\left(\frac{1-\nu}{\theta}\right)n\right]t} ,$$

where $\left[\frac{\sigma(\theta-1)+\rho-(\theta+\nu-1)n}{(1-\alpha)\theta}\right] > 0$ and $\left[\left(\frac{\sigma-\rho}{\theta}\right)-\left(\frac{1-\nu}{\theta}\right)n\right] \equiv \gamma > 0$.

Simple inspection of the last equation suggests that, for given h_0 $\left(\equiv H_0 / L_0 = H_0\right)$, y_t is independent of population size, *L*, along the BGP equilibrium and is proportional to per-capita human capital. ■

Comparing Theorems 3 and 4 of model II with their counterparts of model I (respectively, Theorems 1 and 2) we see that, *ceteris paribus*, two economies (one in which agents choose optimally their own saving rate and the other in which they take their own saving rate as exogenously given) behave similarly as far as the long-run relationship between population (size and growth) and per-capita income (level and growth) are concerned. In particular, in both cases we observe that per-capita income growth and level are independent of population size along the BGP equilibrium (Theorems 2 and 4). Furthermore, population growth is neither necessary nor conducive to long-run growth in per-capita real income (Theorems 1 and 3). The only relevant difference between the two economies is that, contrary to the case where the saving rate is exogenous, when individuals choose endogenously how much to save (model II) the effect of population growth on economic growth can also be neutral (see Theorem 3). This theoretical result fits perfectly with the empirical finding of a weak, even though frequently negative correlation between growth rates of population and income per capita. In the next section we develop a generalization of model II in an attempt to analyze the conditions on the structure of our basic model under which we may observe a positive, as well as a negative or no long-run correlation at all between population and economic growth rates, as empirical evidence seems to suggest most.

4. Population and factor accumulation-based growth with endogenous saving rate and complementarity/substitutability of human and physical capital in the production of new human capital: Model III

Consider the same closed economy of the two previous models where a representative household receives wages and interest income and purchases homogeneous consumption goods produced by competitive firms. The size of the representative household (*L*) rises over time at the constant and exogenous rate $n \geq 0$. The household uses the income it does not consume to accumulate physical capital. Thus, the law of motion of physical capital in per-capita terms ($k_t \equiv K_t / L_t$) still reads as:

$$\dot{k}_t = (r_t - n)k_t + w_t(u_t h_t) - c_t \,,$$

with $c_t\,(\equiv C_t / L_t)$, $u_t h_t\,(\equiv H_{Yt} / L_t)$, r_t and w_t denoting, respectively, per-capita consumption, the fraction of per-capita human capital ($h_t \equiv H_t / L_t$) employed to produce final goods, the real interest rate and the wage per unit of human capital services in production. We also keep on assuming that the stocks of human and physical capital are not subject to any material depreciation. Consumption goods (Y), whose price is set equal to one, are produced competitively through the same aggregate technology we have been using in the previous two models, that is:

$$Y_t = K_t^{\alpha}(H_{Yt})^{1-\alpha}\,, \qquad \alpha \in (0;1).$$

Maximization of the representative firm's profit implies that each input (K_t and H_{Yt}) receives in equilibrium its own marginal product. Hence, with $H_{Yt} \equiv u_t H_t$, we have:

$$r_t \equiv \frac{\partial Y_t}{\partial K_t} = \alpha\left(\frac{H_{Yt}}{K_t}\right)^{1-\alpha} = \alpha\left(\frac{u_t h_t}{k_t}\right)^{1-\alpha} \qquad (28)$$

$$w_t \equiv \frac{\partial Y_t}{\partial H_{Yt}} = (1-\alpha)\left(\frac{K_t}{H_{Yt}}\right)^{\alpha} = (1-\alpha)\left(\frac{k_t}{u_t h_t}\right)^{\alpha}. \qquad (29)$$

Each individual in the economy uses the fraction $(1-u_t)$ of her/his stock of human capital to produce new human capital. The only difference with respect to previous models of this paper consists in introducing physical capital into the production of human capital in such a way that these two factor-inputs can be either substitutes or complementary for each other in the education sector. More formally, at the economy-wide level, the law of motion of aggregate human capital (H_t) is now postulated to be:

$$\dot{H}_t = \sigma(1-u_t)H_t - (\varepsilon\gamma_{Kt})H_t\,, \qquad H_0 > 0\,, \qquad \sigma > 0\,, \qquad (1+\varepsilon) > 0\,, \qquad \varepsilon \neq 0\,, \qquad (30)$$

with σ and ε representing two technological parameters. The first (σ) is the productivity of human capital in the production of new human capital and is positive, while the second (ε) reflects the impact of the growth rate of the aggregate physical capital stock ($\gamma_{Kt} \equiv \dot{K}_t / K_t$), a measure of *learning-by-using* the new technology embodied in new capital goods,[9] on the accumulation of *H*. For given $\sigma > 0$ and $0 < u_t < 1$, the constraint $\varepsilon > -1$ prevents the growth rate of the model's aggregate variables from either exploding ($\varepsilon = -1$) or being negative ($\varepsilon < -1$) along a BGP equilibrium where the ratio of the two endogenous state variables, H_t / K_t, remains invariant. On the other hand, the restriction $\varepsilon \neq 0$ ensures that our aggregate human capital supply function differs significantly from the one initially introduced by Uzawa (1965) and Lucas (1988).[10] Indeed, simple inspection of equation (30) suggests that, with $\varepsilon \neq 0$, the main difference with respect to these two pioneering works consists in postulating a technology for human capital formation in which a faster *learning-by-using* (higher γ_{Kt}) may, *ceteris paribus*, either accelerate ($-1 < \varepsilon < 0$) or slow down ($\varepsilon > 0$) the rate at which human capital accumulates over time, γ_{Ht}.

Equation (30) is borrowed from Alvarez Albelo (1999), with two major differences. The first consists in the fact that in our model the variable measuring the degree of *"learning-by-using the new technology embodied in new capital goods"* (hereafter *"learning-by-using"* or simply *"learning"*) is given by the growth rate of physical capital, and not by net investment in physical capital ($\dot{K}_t$). Thus, we use a relative (rather than absolute) measure of learning. The reason for this is mainly technical. In her original paper, Alvarez Albelo (1999) uses a technology of human capital accumulation that is additive in two components (formal education and learning)[11] and defines the long-run equilibrium of the model (what she calls *steady state*) as a situation where *u* remains constant and the main variables depending on time grow at a constant rate.[12] If we applied both her definition of *learning* ($\dot{K}_t$) and of steady state equilibrium to our framework, we would write $\dot{H}_t = \sigma(1-u)H_t - \left(\varepsilon \dot{K}_t\right)H_t$ and obtain that in the long-run (when *t* goes to infinity) growth in physical capital ceases, that is $\lim_{t \to \infty} \dot{K}_t / K_t \to 0$. Instead, using a balanced growth path equilibrium perspective, equation (30) allows us analyzing the predictions of a two-sector endogenous growth model where in the long-run *u* may continue to stay constant, but the common (and constant) growth rate of all variables (both aggregate and in per-capita terms) is strictly positive. In the next section we define and characterize more formally the BGP equilibrium of this economy.

The second difference, instead, is more substantial and has to do with the fact that, depending on the sign of ε, learning can contribute either to depreciate or to appreciate the existing stock of human capital. More precisely, and unlike Alvarez Albelo (1999) that focuses only on the case where physical and human capital

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[9] Alvarez Albelo (1999).

[10] See Barro and Sala-i-Martin (2004, p. 251, eq. 5.21), with a depreciation rate of human capital equal to zero ($\delta = 0$).

[11] Namely, $\dot{H}_t = \beta H_t (1-U_t) + \gamma \dot{K}_t$, with $\beta, \gamma \in (0;1)$.

[12] *"…The steady state paths are such that* c_t, k_t, h_t *grow at a constant rate and u remains constant. From now on, the variables during the transition will be written in capitals and in the steady state in lowercase. It is easy to check that* c_t, k_t, h_t, y_t *grow at the same rate"* (p. 359). In her model there is no population growth and all variables are in per-capita terms.

are *complementary* for each other in the production of new human capital,[13] we do not make any *a-priori* assumption in this respect. Therefore, equation (30) above represents a generalization of her aggregate technology for human capital production. Indeed, it is today well-recognized (see, among others, Galor and Moav, 2002, p.1148) that the time required for learning the latest technology increases with the rate of technical progress. Thus, and especially for those industries experiencing rapid advancements in technological progress, the presence of large time costs from learning the use of the most up-to-date technology embodied in new capital goods leads to a faster depreciation of the available human capital stock ("*erosion effect*"). An example is the (time) cost of understanding/learning the instructions for the use of new machines. After this cost has been borne, the value of the older knowledge (human capital) can result substantially reduced. In this case we would say that *learning* acts as a mechanism of *endogenous* depreciation of human capital in the equation of skill-supply. This is exactly what happens in equation (30) when $\varepsilon > 0$. As a matter of fact, in this case it is possible to see that, for each $H_t > 0$, physical (K_t) and human capital (H_t) are *substitutes* for each other in the production of new human capital, in the sense that in the long period (when γ_K and γ_H are constant exponential rates) an increase in the learning variable (γ_K), and thus in K_t, always harms investment in human capital ($\dot{H}_t$) and its growth rate (γ_H), ultimately leading to a simultaneous fall of H_t. On the other hand, when $-1 < \varepsilon < 0$, learning acts as a mechanism of *endogenous* appreciation of human capital in the equation of skill-supply and we are able to recover in this case the same situation already analyzed by Alvarez Albelo (1999). Indeed, with $H_t > 0$, now physical (K_t) and human capital (H_t) are *complementary* for each other in the production of new human capital, in the sense that in the long run an increase in the learning variable (γ_K), and thus in K_t, stimulates investment in human capital ($\dot{H}_t$), its growth rate (γ_H), and eventually leads to a simultaneous rise of H_t.[14] Given $\dot{H}_t$, the law of motion of human capital in per-capita terms is given by:

$$\dot{h}_t = \sigma\left(1-u_t\right)h_t - \left(\varepsilon\gamma_{Kt} + n\right)h_t \,. \qquad (30')$$

With a constant intertemporal elasticity of substitution (*CIES*) individual instantaneous utility function, the representative household solves:

$$\underset{\{c_t, u_t, k_t, h_t\}_{t=0}^{\infty}}{\text{Max}} \ \text{U}_0 \equiv \int_0^{\infty} \left(\frac{c_t^{1-\theta} - 1}{1-\theta} \right) e^{-(\rho - \nu n)t} dt \,, \qquad \nu \in [0;1]; \qquad \rho > \nu n \geq 0\,; \qquad \theta > 1 \qquad (31)$$

s.t.: $$\dot{k}_t = (r_t - n)k_t + w_t(u_t h_t) - c_t \,, \qquad n \geq 0 \qquad u_t \in [0;1] \qquad (32)$$

$$\dot{h}_t = \sigma\left(1-u_t\right)h_t - \left(\varepsilon\gamma_{Kt} + n\right)h_t \,, \qquad \sigma > 0\,; \qquad (1+\varepsilon) > 0\,; \qquad \varepsilon \neq 0 \qquad (30')$$

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13 *"…Note that this new technology implies that both types of capital are complementary for each other in such a way that if U_t were equal to unity we would obtain the AK model"* (Alvarez Albelo, 1999, p.358).

14 Sequeira and Reis (2006) and Bucci (2008) also use a technology for human capital accumulation similar to (30) in studying the behaviour of economies where horizontal R&D activity, employing human capital, is the source of endogenous technical progress.

$$\lim_{t\to\infty}\lambda_{kt}k_t = 0\,; \qquad \lim_{t\to\infty}\lambda_{ht}h_t = 0 \tag{33}$$

$k_0 > 0$, $h_0 > 0$.

Households are assumed to be competitive in that, in solving for the optimal paths of the share of skills to be devoted to production activity, $\{u_t\}_{t=0}^{\infty}$, and of per-capita consumption, per-capita physical capital and per-capita human capital, $\{c_t, k_t, h_t\}_{t=0}^{\infty}$, each takes the growth rate of aggregate physical capital (our *learning* variable, γ_{Kt}) as given. The first order conditions of the problem stated above are:

(34) $\dfrac{e^{-(\rho-\nu n)t}}{c_t^{\theta}} = \lambda_{kt}$; (35) $\lambda_{kt} = \dfrac{\sigma}{w_t}\lambda_{ht}$;

(36) $\lambda_{kt}(r_t - n) = -\dot{\lambda}_{kt}$; (37) $\lambda_{kt}u_t w_t + \lambda_{ht}\left[\sigma(1-u_t) - \left(\varepsilon\gamma_{Kt} + n\right)\right] = -\dot{\lambda}_{ht}$.

4.1 Balanced Growth Path (BGP) equilibrium

The BGP equilibrium of this economy is defined as follows.

Definition: *Balanced Growth Path* (BGP) *equilibrium*

A BGP equilibrium is a long-run equilibrium in which: (i) *all variables depending on time grow at a constant (possibly positive) exponential rate;* (ii) *the ratio of the two endogenous state-variables (* H_t / K_t *) remains invariant over time.*

This definition implies that *u* (see 30'), *r* (see 28) and *w* (see 29), are all constant along the BGP equilibrium. It is possible to show that along the BGP equilibrium the following results do hold (mathematical derivation is in appendix B):

$$u = 1 - \frac{(1+\varepsilon)[(\sigma-\rho)-(1-\nu-\theta)n]}{\sigma(\theta+\varepsilon)} \tag{38}$$

$$1-u = \frac{(1+\varepsilon)\left[(\sigma-\rho)-(1-\nu-\theta)n\right]}{\sigma(\theta+\varepsilon)} \tag{39}$$

$$\gamma_c = \gamma_k = \gamma_h = \gamma_y \equiv \gamma = \left(\frac{\sigma-\rho}{\theta+\varepsilon}\right) - \left(\frac{1+\varepsilon-\nu}{\theta+\varepsilon}\right)n \tag{40}$$

$$\frac{h_t}{k_t} = \left\{\frac{\sigma\theta + \varepsilon[\rho + (1-\nu-\theta)n]}{\alpha(\theta+\varepsilon)}\right\}^{1/(1-\alpha)} \cdot \left\{\frac{\sigma(\theta+\varepsilon)}{\sigma(\theta-1)+(1+\varepsilon)[\rho+(1-\nu-\theta)n]}\right\} \tag{41}$$

PROPOSITION 3

Under our assumptions on parameter values, the following restrictions

$$\sigma > \rho + (1+\varepsilon-\nu)n \geq \rho + (1-\nu-\theta)n > \frac{\varepsilon\left[(\nu+\theta-1)n-\rho\right]}{\theta}$$

$$\sigma > \rho \qquad (42)$$
$$\rho - vn > n(\theta - 1)$$

guarantee simultaneously that:

- *the equilibrium growth rate of real per capita income (γ) is positive;*
- *the equilibrium ratio of human to physical capital (h_t / k_t) is also positive;*
- *the equilibrium distribution of human capital across sectors has an interior solution, i.e.* $u \in (0;1)$.

Proof

The proof of Proposition 3 follows immediately from equations (38), (40) and (41) and the restrictions on the parameters stated in equations (31), (32) and (30'). ■

Notice that in (42) we still require $\sigma > \rho$. As already mentioned, this is standard in this class of models. The next theorem analyzes the effects of population growth on real per-capita income growth when $\varepsilon \neq 0$ and $\nu \in (0;1)$. Later on we shall examine separately what happens in other possible cases.

THEOREM 5

Assume that $\varepsilon \neq 0$ *and* $\nu \in (0;1)$. *In an economy that accumulates physical and human capital, where the saving rate is endogenous, economic growth is driven solely by human capital investment, and in which human and physical capital can be either complementary or substitutes to each other in the production of new human capital, real per-capita income growth (* γ *) depends positively on a constant (i.e., a positive combination of preference –* ρ *and* θ *– and technological –* σ *and* ε *– exogenous parameters). The effect of population growth on economic growth crucially depends on whether* ε *(reflecting the impact of learning on the accumulation of human capital) is below, above, or equal to a threshold level,* $\bar{\varepsilon} \equiv -(1-\nu)$. *Given our assumptions on parameter values, economic growth is positive even in the presence of a stable population (n=0).*

Proof

See equation 40. This equation also suggests that $\partial\gamma / \partial n$ is positive when $-1 < \varepsilon < \bar{\varepsilon} < 0$, negative when $\varepsilon > \bar{\varepsilon}$ and equal to zero when $\varepsilon = \bar{\varepsilon}$. ■

In order to see the intuition behind the theorem, it is possible to recast the growth rate of per-capita income in equation (40) as:

$$\gamma = \underbrace{\frac{\sigma(1-u)}{(1+\varepsilon)}}_{\equiv \gamma_K = \gamma_H} - n \, .$$

Thus, the impact of population growth on per-capita income growth can be decomposed into two separate effects: the usual *negative dilution effect* (the term $-n$ above) and the *positive accumulation effect* - the term $\sigma(1-u)/(1+\varepsilon)$. The theorem states that when $-1 < \varepsilon < \bar{\varepsilon} < 0$, the positive accumulation effect prevails over the negative dilution effect and $\frac{\partial \gamma}{\partial n} > 0$; when $\varepsilon > \bar{\varepsilon}$, the negative dilution effect outweighs the positive accumulation effect and $\frac{\partial \gamma}{\partial n} < 0$; finally, when $\varepsilon = \bar{\varepsilon}$ the two effects cancel out each other and $\frac{\partial \gamma}{\partial n} = 0$.

Recalling that in our model human and physical capital are *complements/substitutes* for each other in the production of new human capital respectively when $-1 < \varepsilon < 0$ and $\varepsilon > 0$, the previous theorem leads to:

LEMMA

- *If human and physical capital are complements for each other in the production of new human capital (i.e., $-1 < \varepsilon < 0$), the effect of population growth on real per-capita income growth is ambiguous:* $\frac{\partial \gamma}{\partial n} \gtreqless 0$;
- *If human and physical capital are substitutes for each other in the production of new human capital (i.e., $\varepsilon > 0$), the effect of population growth on real per-capita income growth is always negative:* $\frac{\partial \gamma}{\partial n} < 0$.

Proof:
When $-1 < \varepsilon < \bar{\varepsilon}$ human and physical capital are *complements* and $\frac{\partial \gamma}{\partial n} > 0$.
When $\varepsilon = \bar{\varepsilon}$ human and physical capital are *complements* and $\frac{\partial \gamma}{\partial n} = 0$.
When $\bar{\varepsilon} < \varepsilon < 0$ human and physical capital are *complements* and $\frac{\partial \gamma}{\partial n} < 0$.
When $\varepsilon > 0$ human and physical capital are *substitutes* and $\frac{\partial \gamma}{\partial n} < 0$. ■

The intuition is as follows. The growth rate of per-capita income can also be recast as:

$$\gamma = \sigma(1-u) - \varepsilon\gamma_K - n,$$

where $\gamma_K = \frac{(\sigma-\rho)-(1-\nu-\theta)n}{(\theta+\varepsilon)}$ is a positive function of *n*. For given *u* and assuming that human and physical capital are substitutes for each other in the production of new human capital ($\varepsilon > 0$), an increase in population size (an increase of *n*) always exerts a negative effect (both direct and indirect, through learning) on real per-capita income growth. Instead, when human and physical capital are complements for each other ($-1 < \varepsilon < 0$), as long as population rises the term $-\varepsilon\gamma_K$ becomes increasingly positive in the expression above. Thus, in this case the whole effect of population growth on real per-capita income growth may be either positive, or negative, or else equal to zero.

THEOREM 6

In an economy that accumulates physical and human capital, where the saving rate is endogenous, economic growth is driven solely by human capital investment, and in which human and physical capital can be either complementary or substitutes to each other in the production of new human capital, both the growth rate

(γ) and the level (y_t) of real per-capita income are independent of population size along the BGP equilibrium.

Proof

Equation (40) reveals that per-capita income growth (γ) depends on population growth (*n*), but not on population size (*L*). From (3), along the BGP equilibrium per-capita real income can also be written as:

$$y_t \equiv \frac{Y_t}{L_t} = \left(\frac{h_t}{k_t}\right)^{-\alpha} u^{1-\alpha} h_t \,,$$

where *u* is the constant share of the available stock of human capital devoted to production of goods. Plugging (38) and (41) into the equation above yields:

$$y_t = \left\{\frac{\alpha(\theta+\varepsilon)}{\sigma\theta+\varepsilon\left[\rho+(1-\nu-\theta)n\right]}\right\}^{\frac{\alpha}{1-\alpha}} \left\{\frac{\sigma(\theta-1)+(1+\varepsilon)\left[\rho+(1-\nu-\theta)n\right]}{\sigma(\theta+\varepsilon)}\right\} h_0 e^{\left[\left(\frac{\sigma-\rho}{\theta+\varepsilon}\right)-\left(\frac{1+\varepsilon-\nu}{\theta+\varepsilon}\right)n\right]t} = \Lambda h_0 e^{\gamma t} \,,$$

where:

$$\Lambda \equiv \left\{\frac{\alpha(\theta+\varepsilon)}{\sigma\theta+\varepsilon\left[\rho+(1-\nu-\theta)n\right]}\right\}^{\frac{\alpha}{1-\alpha}} \left\{\frac{\sigma(\theta-1)+(1+\varepsilon)\left[\rho+(1-\nu-\theta)n\right]}{\sigma(\theta+\varepsilon)}\right\} \quad \text{and}$$

$$\gamma_h \equiv \gamma = \left(\frac{\sigma-\rho}{\theta+\varepsilon}\right)-\left(\frac{1+\varepsilon-\nu}{\theta+\varepsilon}\right)n$$

are both positive, according to our assumptions on the model's parameter values. Simple inspection of the last equation suggests that, for given h_0 $\left(\equiv H_0 / L_0 = H_0\right)$, y_t is independent of population size, *L*, along the BGP equilibrium and is proportional to per-capita human capital. ■

The last theorem analyses the relationship between population and economic growth rates in two special cases.

THEOREM 7

In an economy that accumulates physical and human capital, where the saving rate is endogenous, economic growth is driven solely by human capital investment, and in which human and physical capital can be either complementary or substitutes to each other in the production of new human capital, the effect of population growth (n) on the growth rate of real per-capita income (γ) is:

- *Unambiguously equal to zero when $\varepsilon = 0$ and $\nu = 1$ are simultaneously checked;*
- *Unambiguously negative when $\varepsilon = 0$ and $\nu \in [0;1)$.*

Proof

When $\varepsilon = 0$ and $\nu = 1$, equation (40) gives: $\gamma = (\sigma-\rho)/\theta$. Clearly, in this case the growth rate of real per-capita income is totally independent of population growth (*n*). On the other hand, when $\varepsilon = 0$ and $\nu \in [0;1)$ we have: $\gamma = \left[\sigma-\rho-(1-\nu)n\right]/\theta$, with population growth now affecting negatively economic growth. ■

With respect both to these particular cases (including the basic Uzawa, 1965 and Lucas, 1988 models[15]) and to the previous two models of this paper, the model presented in this section has demonstrated that when human and physical capital interact with each other in such a way that they can be either substitutes or complementary in the production of new human capital, then the effect of demographic change on per-capita income growth is no longer obvious, depending not only on whether these two types of capital are complements or substitutes in the education sector, but also on their degree of complementarity.

5. Concluding Remarks

Using a balanced growth path equilibrium perspective, in this paper we built a simple two-sector endogenous growth model with human and physical capital accumulation in order to analyze whether it is possible to recover the most important and recent results on the long run connection between population and real per-capita income growth rates of the empirical as well as theoretical growth literature (either the one dealing solely with endogenous technological change or the one dealing jointly with endogenous technological change *and* human capital accumulation). At this aim we described three different economic environments, sharing the same basic assumptions as well as the source of economic growth (*i.e.*, human capital accumulation). Our conclusion from the first two models is that when individuals can choose endogenously how much to save, then economic growth can be either negatively affected by or completely independent of population growth. The negative impact of population growth on economic growth is due to the traditional *dilution effect*, that is the cost (in terms of reproducible capital) of equipping the new workers in the population with the existing capital-labor ratio. Instead, the lack of correlation between economic and population growth can be explained by the structure of agents' preferences and, in particular, by the presence of a degree of agents' *altruism* towards future generations exactly equal to one in the inter-temporal aggregate utility of the representative household (*time-preference effect*).

However, it seems to be a fair conclusion of the huge body of empirical research conducted by demographers and economists that, depending on the country, population change may either contribute, or deter, or even play no impact on economic development. Thus, in the last part of the paper we addresses the question of whether (and, if yes, under which conditions) our model would be able to account jointly for a positive, negative, or no effect at all of population growth on economic growth. Contrary to existing literature, in the third model we assumed that learning (measured by the growth of aggregate physical capital) affects agents' decision of how much to invest in formal education. Furthermore, we did not do any *a-priori* hypothesis on the relationship of complementarity/substitutability between physical and human capital in the production of new human capital. The presence of the growth rate of physical capital in the law of motion of skills allowed us to introduce in the model a sort of *endogenous* mechanism of depreciation (or appreciation) of embodied knowledge. In this framework, we found that population growth can play a positive, negative, or else no role on economic growth depending on the degree of

[15] See Barro and Sala-i-Martin (2004), equation 5.31, p. 252 with $B \equiv \sigma$ and $\delta = 0$.

complementarity/substitutability between physical capital and formal education in the process of agents' skill acquisition. The intuition goes as follows. *Ceteris paribus* (in particular, for a given per-capita physical capital stock), an increase in population size always raises the growth rate of aggregate physical capital. If human and physical capital are substitutes for each other in the production of new human capital, the joint increase of population and the physical capital growth rate definitely lowers human capital accumulation at the individual level and, thus, per-capita income growth. Instead, if human and physical capital are complements, as long as population and the growth rate of aggregate physical capital increase, the rate of per-capita human capital accumulation (and, thus, the rate of per-capita income growth) may either go up, down, or else be exactly equal to zero.

For future theoretical research it would be interesting to analyze how the results of this paper might ultimately change either in the presence of two forms of capital (human and physical capital) but endogenous population growth (endogenous fertility) or in presence of still exogenous population growth but with three forms of capital (human, physical and technological capital).

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Appendix A (not to be published)

In this appendix we prove the set of results (23)-(26) in the main text. Combining (20) and (22) in the text and using (21) yields, respectively:

A1) $$\frac{\dot{\lambda}_{ht}}{\lambda_{ht}} = -(\sigma - n)$$

A2) $$\frac{\dot{\lambda}_{kt}}{\lambda_{kt}} = -(r_t - n).$$

Equation (20) also implies:

A3) $$\frac{\dot{\lambda}_{kt}}{\lambda_{kt}} = \frac{\dot{\lambda}_{ht}}{\lambda_{ht}} - \gamma_w, \qquad \text{where } \gamma_w \equiv \frac{\dot{w}_t}{w_t}.$$

Plugging (A1) and (A2) into (A3) yields:

A4) $$r_t = \sigma + \gamma_w.$$

With an aggregate production function

$$Y_t = K_t^{\alpha} (H_{Yt})^{1-\alpha},$$

it is immediate to obtain:

A5) $$r_t \equiv \frac{\partial Y_t}{\partial K_t} = \alpha \left(\frac{u_t h_t}{k_t} \right)^{1-\alpha}.$$

A6) $$w_t \equiv \frac{\partial Y_t}{\partial H_{Yt}} = (1-\alpha)\left(\frac{k_t}{u_t h_t} \right)^{\alpha},$$ where we used the definitions $H_{Yt} \equiv u_t H_t$, $h_t \equiv \frac{H_t}{L_t}$ and $k_t \equiv \frac{K_t}{L_t}$.

Along the BGP equilibrium, u and the ratio of the two endogenous state variables in per-capita terms (h_t / k_t) are constant. Hence, and using (A5) and (A6), it follows that along the BGP equilibrium r and w are constant. In turn, this implies:

A7) $$\frac{\dot{w}_t}{w_t} \equiv \gamma_w = 0$$

A8) $$r_t = \sigma + \gamma_w = \sigma.$$

With perfectly-competitive capital markets, the rates of return (in terms of goods) of physical and human capital (respectively, r and w) are equal in the long-run. Equating (A8) and (A6) leads to:

A9) $$u\left(\frac{h_t}{k_t} \right) = \left(\frac{1-\alpha}{\sigma} \right)^{1/\alpha}.$$

Using (19) in the main text, (A2) and (A8) we get:

A10) $$\frac{\dot{c}_t}{c_t} \equiv \gamma_c = \frac{1}{\theta}\left[\sigma - \rho - (1-\nu) n \right].$$

From (15), (A8) and (A2):

A11) $\frac{\dot{\lambda}_{kt}}{\lambda_{kt}} = -\gamma_k + wu\frac{h_t}{k_t} - \frac{c_t}{k_t}$, $\gamma_k \equiv \frac{\dot{k}_t}{k_t}$.

Instead, by combining (16') and (A1):

A12) $\frac{\dot{\lambda}_{ht}}{\lambda_{ht}} = -\gamma_h - \sigma u$, $\gamma_h \equiv \frac{\dot{h}_t}{h_t}$.

Since $\gamma_w = 0$, from (A3) we get:

A13) $\frac{\dot{\lambda}_{kt}}{\lambda_{kt}} = \frac{\dot{\lambda}_{ht}}{\lambda_{ht}} \Rightarrow -\gamma_k + wu\frac{h_t}{k_t} - \frac{c_t}{k_t} = -\gamma_h - \sigma u$.

Along the BGP equilibrium h_t / k_t is constant, implying that $\gamma_h = \gamma_k$. Therefore, (A13) leads to:

A14) $\frac{c_t}{k_t} = wu\left(\frac{h_t}{k_t}\right) + \sigma u$.

With w, u and $\left(\frac{h_t}{k_t}\right)$ all constant along the BGP equilibrium, equation (A14) implies that $\frac{c_t}{k_t}$ is constant, as well. Thus, we can conclude that along the BGP equilibrium it must be true that:

(A15) $\gamma_c = \gamma_k = \gamma_h = \gamma_y \equiv \gamma = \sigma(1-u) - n$.

To find out $(1-u)$ and u, we equate (A15) with (A10) and obtain:

(A16) $(1-u) = \frac{(\sigma - \rho) + (\theta + \nu - 1)n}{\sigma\theta}$

(A17) $u = \frac{\sigma(\theta - 1) + \rho - (\theta + \nu - 1)n}{\sigma\theta}$.

Plugging (A16) into (A15) yields:

(A18) $\gamma_c = \gamma_k = \gamma_h = \gamma_y \equiv \gamma = \sigma(1-u) - n = \left(\frac{\sigma - \rho}{\theta}\right) - \left(\frac{1-\nu}{\theta}\right)n$.

Finally, from (A9):

(A19) $\left(\frac{h_t}{k_t}\right) = \left(\frac{1-\alpha}{\sigma}\right)^{1/\alpha} \cdot \frac{1}{u} = \left(\frac{1-\alpha}{\sigma}\right)^{1/\alpha} \cdot \left[\frac{\sigma\theta}{\sigma(\theta-1) + \rho - (\theta + \nu - 1)n}\right]$.

In the end of this appendix we want to find out the restriction on parameter values under which the two transversality conditions:

$$\lim_{t\to\infty} \lambda_{kt} k_t = 0 \quad \text{and} \quad \lim_{t\to\infty} \lambda_{ht} h_t = 0,$$

are checked. With $\sigma > 0$ and the (given) initial values (at *t=0*) of the two state variables (k_0 and h_0) and their respective shadow prices (λ_{k0} and λ_{h0}) also positive, equations (A1), (A2), (A8) and (A15) allow us concluding that the transversality conditions are satisfied when $\sigma > \frac{-\rho + (\theta + \nu - 1)n}{(\theta - 1)}$, which is reported in Proposition 2 in the main text. ■

Appendix B (not to be published)

In this appendix we prove the set of results (38)-(41) in the main text. Along the BGP equilibrium all variables depending on time grow at a constant exponential rate and, thus, *u* is constant as well. Combining (35) and (37) and using (36) yields, respectively:

B1) $$\frac{\dot{\lambda}_{ht}}{\lambda_{ht}} = -\left[\sigma - \left(\varepsilon\gamma_K + n\right)\right], \qquad \gamma_K \equiv \frac{\dot{K}_t}{K_t}$$

B2) $$\frac{\dot{\lambda}_{kt}}{\lambda_{kt}} = -\left(r_t - n\right).$$

Equation (35) also implies:

B3) $$\frac{\dot{\lambda}_{kt}}{\lambda_{kt}} = \frac{\dot{\lambda}_{ht}}{\lambda_{ht}} - \gamma_w, \qquad \gamma_w \equiv \dot{w}_t / w_t .$$

Plugging (B2) and (B1) into (B3) yields:

B4) $$r_t = \left(\sigma - \varepsilon\gamma_K\right) + \gamma_w .$$

From 28 and 29 in the main text it follows immediately that along the BGP (where *k* and *h* grow at the same rate $\gamma_k = \gamma_h \equiv \gamma$), *r* and *w* are constant ($\gamma_w = 0$). Therefore:

B5) $$r_t = \sigma - \varepsilon\gamma_K = \sigma - \varepsilon\left(\gamma + n\right) = r , \qquad k_t \equiv K_t / L_t .$$

Equalization of (B5) and 28 leads to:

B6) $$\left(\frac{uh_t}{k_t}\right) = \left[\frac{\sigma - \varepsilon\left(\gamma + n\right)}{\alpha}\right]^{1/1-\alpha} = \left(\frac{r}{\alpha}\right)^{1/1-\alpha} .$$

Using 34, (B2) and (B5):

B7) $$\frac{\dot{c}_t}{c_t} \equiv \gamma_c = \frac{1}{\theta}\left[\left(\sigma - \rho\right) - \varepsilon\gamma - \left(1 + \varepsilon - \nu\right)n\right].$$

From 32 and (B2) one obtains:

B8) $$\frac{\dot{\lambda}_{kt}}{\lambda_{kt}} = -\gamma + wu\frac{h_t}{k_t} - \frac{c_t}{k_t} .$$

Instead, by combining 30’ and (B1), we get:

B9) $$\frac{\dot{\lambda}_{ht}}{\lambda_{ht}} = -\gamma - \sigma u .$$

Since $\gamma_w = 0$, from (B3) it is first of all possible to obtain:

B10) $$\frac{\dot{\lambda}_{kt}}{\lambda_{kt}} = \frac{\dot{\lambda}_{ht}}{\lambda_{ht}} ,$$

and, then, by equating (B8) and (B9):

B11) $\frac{c_t}{k_t} = wu\left(\frac{h_t}{k_t}\right) + \sigma u$.

With w, u and h_t / k_t being constant along the BGP equilibrium, (B11) together with 1' and 30' implies:

$$\gamma_c = \gamma_k = \gamma_h = \gamma_y \equiv \gamma = \sigma(1-u) - \varepsilon\gamma_K - n \qquad \Rightarrow$$

B12) $\gamma = \frac{\sigma(1-u)}{(1+\varepsilon)} - n$, $\qquad \gamma_K = \gamma + n$.

With $\gamma_c \equiv \gamma$, from (B7) it easily derives that:

B13) $\gamma = \frac{(\sigma-\rho)}{(\theta+\varepsilon)} - \frac{(1+\varepsilon-\nu)}{(\theta+\varepsilon)}n$.

To find out ($1-u$), we equate (B12) and (B13) and get:

B14) $1-u = \frac{(1+\varepsilon)\left[(\sigma-\rho)-(1-\nu-\theta)n\right]}{\sigma(\theta+\varepsilon)}$.

Combining (B5) and (B13):

B15) $r = \sigma - \varepsilon(\gamma+n) = \frac{\sigma\theta + \varepsilon\left[\rho + (1-\nu-\theta)n\right]}{(\theta+\varepsilon)}$.

Given *r* and *u*, from (B6) it follows:

B16) $\left(\frac{h_t}{k_t}\right) = \left(\frac{r}{\alpha}\right)^{1/1-\alpha}\left(\frac{1}{u}\right) = \left\{\frac{\sigma\theta + \varepsilon[\rho + (1-\nu-\theta)\cdot n]}{\alpha(\theta+\varepsilon)}\right\}^{1/(1-\alpha)} \cdot \left\{\frac{\sigma(\theta+\varepsilon)}{\sigma(\theta-1)+(1+\varepsilon)[\rho+(1-\nu-\theta)\cdot n]}\right\}$.

Using (B10) and (B9), it is possible to recast the two transversality conditions as:

$$\lim_{t\to\infty}\lambda_{kt}k_t = (\lambda_{k0}k_0)\lim_{t\to\infty}e^{-(\sigma u)t} = 0 \qquad \text{and} \qquad \lim_{t\to\infty}\lambda_{ht}h_t = (\lambda_{h0}h_0)\lim_{t\to\infty}e^{-(\sigma u)t} = 0 ,$$

where k_0 , h_0 , λ_{k0} and λ_{h0} are, respectively, the (given) initial values (at *t=0*) of the two state-variables (*k* and *h*) and their respective shadow prices (λ_k and λ_h). With $\sigma > 0$, such conditions are trivially satisfied whenever $u \in (0;1)$. ■