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POVERTY RANKINGS OF OPPORTUNITY PROFILES

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1. Introduction

Poverty reduction plays a prominent role in political debates in many countries. Methods and techniques to make poverty comparisons are necessary tools in order to design and to evaluate policies aimed at poverty reduction.

Since the publication of Sen's (1976) pioneering paper on poverty measurement, in the last quarter century a great deal has been written on this subject. Several measures of poverty, including the one suggested by Sen (1976), are now available in the literature. However, in most of the existing literature, income or consumption expenditures has been regarded as the only relevant dimension of poverty. But poverty is essentially a multidimensional phenomenon and income or consumpition is just one indicator. The necessity to move from an income-based evaluation of social inequities towards a more comprehensive domain has been argued, among other, by Rawls (1971), Sen (1980), Roemer (1996). These last authors, specifically, have argued in favour of opportunities as the proper *space* for distributive judgments. An individual's opportunities are described by a set rather than by a scalar, as it is the case with income poverty. As a consequence, the problem becomes that of ranking different distributions of opportunity sets.

The question of how to rank different opportunity distributions has been first addressed by Kranich (1996), who however focused only on inequality rankings. There is now an extensive literature concerned with the measurement of inequality of opportunity: see, for example, Arlegi and Nieto [1], Bossert, Fleurbaey, and Van de gaer [6], Herrero [10], Herrero, Iturbe-Ormaetxe, and Nieto [11], Kranich [14, 15], Ok [16], Ok and Kranich [17], and Savaglio and Vannucci [20]. A survey of this literature may be found in Barbera' et al. [3].

On the other hand, the question of how to rank different distributions of opportunities in terms of the poverty they exhibit has never been addressed before. The present paper fills this gap. We address the problem of ranking profiles of opportunity sets on the basis of poverty[1].

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[1]Our contribution is related to the literature on multidimensional poverty mesurement: see, among others,. Tsui (2002), Chakravarty et al. (1998) and Bourguignon and Chakravarty (1999, 2002). However, the framework we propose is more general.

Our approach is axiomatic. We propose a number of properties that a poverty relation on the possible distributions (profiles) of finite opportunity sets should satisfy and we study their logical implications. We characterize two rankings: the Head-Count and the Opportunity-Gap poverty rankings . These generalize the most widely used poverty measures used in the income poverty framework, namely the *head count ratio* and *the income poverty gap*. In addition, we characterize two lexicographic rankings based on the HC and OG rankings and a third one based on a linear combination of the head-count and gap criteria.

2. The analytical framework

Let $N = \{1, .., n\}$ denote the finite set of relevant population units, X an universal nonempty set of opportunities, and $\mathcal{P}[X]$ the set of all *finite* subsets of X. Elements of $\mathcal{P}[X]$ are referred to as *opportunity sets*, and mappings $\mathsf{Y} = (Y_1, ..., Y_n) \in \mathcal{P}[X]^N$ as profiles of opportunity sets, or simply *opportunity profiles.*

In poverty measurement, a crucial role is played by the poverty threshold or povertiy line. In the income poverty framework, a poverty line is that income level that divides the population into two sets: the poor and the non-poor.

An analog of the poverty line in our fraework is a *poverty threshold,* which is a set $T \in \mathcal{P}[X]$. Any poverty threshold T induces a preorder $\succcurlyeq^*_T$on $\mathcal{P}[X]$ by the following rule: for any $Y, Z \in \mathcal{P}[X]$, $Y \succcurlyeq^*_T Z$ iff $[Y \supseteq Z$ or $Y \supseteq T]$. The notation $\mathsf{Y}_{|T}$ will be employed in the rest of this paper to denote opportunity profile $(Y_i \cap T)_{i \in N}$.

The poverty threshold T identifies a set of *essential alternatives*: an individual is declared as poor if her set does not contain all the essential alternatives, i.e., all the altarnatives contained in T. In this case he is declared to be *below* the proverty treshold.

A *poverty ranking* of opportunity profiles -under threshold T- is a preorder $\succcurlyeq_T$ on $\mathcal{P}[X]^N$ such that for any $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$, $\mathsf{Y} \succcurlyeq_T \mathsf{Z}$ whenever $Z_i \succcurlyeq^*_T Y_i$ for each $i \in N$. We shall focus on poverty rankings with certain additional properties to be specified below.

The *head-count (HC) poverty ranking* -under threshold T- is the preorder $\succcurlyeq^h_T$ on $\mathcal{P}[X]^N$ defined as follows: for any $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$, $\mathsf{Y} \succcurlyeq^h_T \mathsf{Z}$ iff $h_T(\mathsf{Y}) \geq h_T(\mathsf{Z})$ where for each $\mathsf{W} \in \mathcal{P}[X]^N$, $h_T(\mathsf{W}) = \# H_T(\mathsf{W})$, and $H_T(\mathsf{W}) = \{i \in N : W_i \nsupseteq T\}$.

The head-count poverty ordering captures the incidence of poverty: i.e., how many individuals are below the poverty treshold T.

The *opportunity-gap (OG) poverty ranking* -under threshold T- is the preorder $\succcurlyeq^g_T$ on $\mathcal{P}[X]^N$ defined as follows: for any $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$, $\mathsf{Y} \succcurlyeq^h_T \mathsf{Z}$ iff $g_T(\mathsf{Y}) \geq g_T(\mathsf{Z})$, where for each $\mathsf{W} \in \mathcal{P}[X]^N$, $g_T(\mathsf{W}) = \sum_{i \in H_T(\mathsf{W})} \# \{x : x \in T \setminus W_i\}$.

The opportunity-gap poverty ranking captures the aggregate intensity of poverty. For each individual, the intensity of poverty is measured by the number of essential alternatives he does not have access to. Hence, the opportunity-gap tells us how poor are the poor.

3. The axioms

In order to provide characterizations of HC and OG we introduce now the following properties for a poverty ranking $<_T$of $\mathcal{P}[X]^N$.

Axiom 1 (*Anonymity (AN)*). *For any permutation π of N, and any $\mathsf{Y} \in \mathcal{P}[X]^N$: $\mathsf{Y} \sim_T \pi\mathsf{Y}$ (where $\pi\mathsf{Y} = (Y_{\pi(1)}, ..., Y_{\pi(n)})$).*

Axiom 2 (*Irrelevance of Inessential Opportunities (IIO)*). *For any $\mathsf{Y} \in \mathcal{P}[X]^N$, $i \in N$, and $x \in Y_i \setminus T$: $\mathsf{Y} \sim_T (\mathsf{Y}_{-i}, Y_i \setminus \{x\})$.*

Axiom 3 (*Dominance at Essential Profiles (DEP)*). *For any $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$ such that both $\{Y_1, ..., Y_n\} \subseteq \{T, \emptyset\}$ and $\{Z_1, ..., Z_n\} \subseteq \{T, \emptyset\}$, $\mathsf{Y} \succ_T \mathsf{Z}$ iff $\#\{i \in N : Y_i = \emptyset\} > \#\{i \in N : Z_i = \emptyset\}$.*

Axiom 4 (*Irrelevance of Poor's Opportunity Deletions (IPOD)*). *For any $\mathsf{Y} \in \mathcal{P}[X]^N$, $i \in H_T(\mathsf{Y})$, and $x \in Y_i$: $\mathsf{Y} \sim_T (\mathsf{Y}_{-i}, Y_i \setminus \{x\})$.*

Axiom 5 (*Strict Monotonicity with respect to Essential Deletions (SMED)*). *For any $\mathsf{Y} \in \mathcal{P}[X]^N$, $i \in N$, and $x \in Y_i \cap T$: $(\mathsf{Y}_{-i}, Y_i \setminus \{x\}) \succ_T \mathsf{Y}$.*

Axiom 6 (*Independence of Balanced Essential Deletions (IBED)*). *For any $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$, $i \in N$, $y \in Y_i \cap T$ and $z \in Z_i \cap T$: $\mathsf{Y} <_T \mathsf{Z}$ iff $(\mathsf{Y}_{-i}, Y_i \setminus \{y\}) <_T (\mathsf{Z}_{-i}, Z_i \setminus \{z\})$.*

4. The results

We are now able to state our characterizations of the HC and OG rankings.

Proposition 1. *Let $<_T$ be a poverty ranking of $\mathcal{P}[X]^N$ under threshold $T \subseteq X$. Then $<_T$ is the HC ranking $<_T^h$ iff $<_T$ satisfies* AN, IIO, IPOD *and* DEP.

Proof. It is straightforward to check that $<_T^h$ is a poverty ranking and does indeed satisfy AN, IIO, DEP and IPOD.

Conversely, suppose $<_T$ is a poverty ranking that satisfies AN, NT, IIO, and IPOD.

Now, consider $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$ such that $\mathsf{Y} <_T \mathsf{Z}$.Then, by repeated application of IIO and transitivity, $\mathsf{Y}_{|T} <_T \mathsf{Z}_{|T}$.

Next, observe that $(T^{N \setminus H_T(\mathsf{Y})}, \emptyset^{H_T(\mathsf{Y})}) \sim_T \mathsf{Y}_{|T} <_T \mathsf{Z}_{|T} \sim_T (T^{N \setminus H_T(\mathsf{Z})}, \emptyset^{H_T(\mathsf{Z})})$, by repeated application of IPOD. Let us now suppose that $h_T(\mathsf{Z}) > h_T(\mathsf{Y})$: then, by AN and DEP, $\mathsf{Z} \succ_T \mathsf{Y}$, a contradiction. Hence, $h_T(\mathsf{Y}) \geq h_T(\mathsf{Z})$ i.e. $\mathsf{Y} <_T^h \mathsf{Z}$.

To prove the reverse inclusion, suppose that $\mathsf{Y} <_T^h \mathsf{Z}$ i.e. $h_T(\mathsf{Y}) \geq h_T(\mathsf{Z})$. Then, consider $(T^{N \setminus H_T(\mathsf{Y})}, \emptyset^{H_T(\mathsf{Y})})$, $(T^{N \setminus H_T(\mathsf{Z})}, \emptyset^{H_T(\mathsf{Z})})$ and a permutation π of N such that $\pi(H_T(\mathsf{Z})) \subseteq \pi(H_T(\mathsf{Y}))$.

By IIO, $\mathsf{Y} \sim_T (T^{N \setminus H_T(\mathsf{Y})}, \emptyset^{H_T(\mathsf{Y})})$ and $\mathsf{Z} \sim_T (T^{N \setminus H_T(\mathsf{Z})}, \emptyset^{H_T(\mathsf{Z})})$; by AN, $(T^{N \setminus H_T(\mathsf{Y})}, \emptyset^{H_T(\mathsf{Y})}) \sim_T (T^{\pi(N \setminus H_T(\mathsf{Y}))}, \emptyset^{\pi(H_T(\mathsf{Y}))})$ and $(T^{N \setminus H_T(\mathsf{Z})}, \emptyset^{H_T(\mathsf{Z})}) \sim_T (T^{\pi(N \setminus H_T(\mathsf{Z}))}, \emptyset^{\pi(H_T(\mathsf{Z}))})$. Clearly, if $\pi(H_T(\mathsf{Z})) = \pi(H_T(\mathsf{Y}))$ then

$(T^{\pi(N\setminus H_T(\mathsf{Y}))},\emptyset^{\pi(H_T(\mathsf{Y}))}) = (T^{\pi(N\setminus H_T(\mathsf{Z}))},\emptyset^{\pi(H_T(\mathsf{Z}))})$ hence, by transitivity of $\succcurlyeq_T$, $\mathsf{Y} \sim_T \mathsf{Z}$. Let us then suppose that $\pi(H_T(\mathsf{Z})) \subset \pi(H_T(\mathsf{Y}))$. By DEP, it follows that
$(T^{\pi(N\setminus H_T(\mathsf{Y}))},\emptyset^{\pi(H_T(\mathsf{Y}))}) \succ_T (T^{\pi(N\setminus H_T(\mathsf{Z}))},\emptyset^{\pi(H_T(\mathsf{Z}))})$
hence in particular $\mathsf{Y} \succcurlyeq_T \mathsf{Z}$. ¤

Remark 1. *The characterization provided above is tight. To check the validity of this claim, consider the following examples.*

i) To begin with, consider the non-anonymous refinement of HC defined by the following rule: $\mathsf{Y} \succcurlyeq_T^{h_1} \mathsf{Z}$ *iff i)* $\mathsf{Y} \succcurlyeq_T^{h} \mathsf{Z}$ *and* $\{Y_i, Z_i\} \subseteq \{T,\emptyset\}$ *for each* $i \in N$ *or ii)* $\mathsf{Y} \succ_T^{h} \mathsf{Z}$ *or iii)* $\mathsf{Y} \sim_T^{h} \mathsf{Z}$*, there exist* $i, j \in N$ *such* $\{Y_i, Z_j\} \cap \{T,\emptyset\} = \emptyset$*, and* $Y_1 \nsupseteq T$*. Clearly,* $\succcurlyeq_T^{h_1}$ *is a poverty ranking that satisfies IIO, IPOD and DEP, but violates AN.*

ii) Consider the refinement of HC defined by the following rule: $\mathsf{Y} \succcurlyeq_T^{h^*} \mathsf{Z}$ *iff* $\mathsf{Y} \succ_T^{h} \mathsf{Z}$ *or* $\mathsf{Y} \sim_T^{h} \mathsf{Z}$ *and* $\#\{i \in N : Y_i \supset T\} \leq \#\{i \in N : Z_i \supset T\}$*. Such a preorder is a poverty ranking that satisfies AN, DEP and IPOD but violates IIO.*

iii) Consider the universal indifference poverty ranking: i.e. $\mathsf{Y} \succcurlyeq^{I} \mathsf{Z}$ *for any* $\mathsf{Y},\mathsf{Z} \in \mathcal{P}[X]^N$*. That ranking does satisfy AN, IIO and IPOD but violates DEP.*

iv) Consider the OG-refinement of HC as defined by the following rule: $\mathsf{Y} \succcurlyeq_T^{h_g} \mathsf{Z}$ *iff either* $\mathsf{Y} \succ_T^{h} \mathsf{Z}$ *or (*$\mathsf{Y} \sim_T^{h} \mathsf{Z}$ *and* $g_T(\mathsf{Y}) \geq g_T(\mathsf{Z})$*). Such a preorder is a poverty ranking that satisfies AN, IIO and DEP, but fails to satisfy IPOD.*

Proposition 2. *Let* $\succcurlyeq_T$ *be a poverty ranking of* $\mathcal{P}[X]^N$ *under threshold* $T \subseteq X$*. Then* $\succcurlyeq_T$ *is the OG ranking* $\succcurlyeq_T^{g}$ *iff* $\succcurlyeq_T$ *satisfies* AN, IIO, SMED *and* IBED*.*

Proof. it is easily checked that $\succcurlyeq_T^{g}$ is a poverty ranking and does satisfy AN, IIO, SMED and IBED.

Conversely, suppose $\succcurlyeq_T$ is a poverty ranking that satisfies AN, IIO, SMED and IBED.

Then, consider $\mathsf{Y},\mathsf{Z} \in \mathcal{P}[X]^N$ such that $\mathsf{Y} \succcurlyeq_T \mathsf{Z}$. Again, by repeated application of IIO and transitivity, $\mathsf{Y}_{|T} \succcurlyeq_T \mathsf{Z}_{|T}$. Now, suppose that $g_T(\mathsf{Z}) > g_T(\mathsf{Y})$. Then, by repeated application of IBED, $\mathsf{Z}'_{|T} \sim_T \mathsf{Y}_{|T}$ for some Z' such that $Z'_i \subseteq Z_i$ for each $i \in N$, and $g_T(\mathsf{Z}') = g_T(\mathsf{Y})$. It follows that, by repeated application of SMED, $\mathsf{Z}_{|T} \succ_T \mathsf{Z}'_{|T}$, hence by transitivity, $\mathsf{Z}_{|T} \succ_T \mathsf{Y}_{|T}$. Thus, by repeated application of IIO and transitivity again, $\mathsf{Z} \succ_T \mathsf{Y}$, a contradiction.

On the other hand, suppose that $\mathsf{Y} \succcurlyeq_T^{g} \mathsf{Z}$ i.e. $g_T(\mathsf{Y}) \geq g_T(\mathsf{Z})$, and consider $\mathsf{T} = (T,...,T) \in \mathcal{P}[X]^N$. Of course, $\mathsf{T} \sim_T \mathsf{T}$, by reflexivity. Then, by AN and repeated application of IBED to $\mathsf{T} \sim_T \mathsf{T}$, it follows that $\mathsf{Y}' \succcurlyeq_T \mathsf{Z}$ for some Y' such that $Y'_i \setminus T = Y_i \setminus T$ and $Y_i \subseteq Y'_i$ for each $i \in N$, and $g_T(\mathsf{Y}') = g_T(\mathsf{Z})$. If in particular $g_T(\mathsf{Y}') = g_T(\mathsf{Y})$ then $\mathsf{Y}' = \mathsf{Y}$ hence $\mathsf{Y} \succcurlyeq_T \mathsf{Z}$, and we are done. Otherwise, there exist $i \in N$ and $x \in T \cap (Y'_i \setminus Y_i)$, hence $\mathsf{Y} \succ_T \mathsf{Z}$ by transitivity and repeated application of SMED. In any case, $\mathsf{Y} \succcurlyeq_T \mathsf{Z}$ as required. ¤

Remark 2. *The foregoing characterization is also tight. To verify that claim consider the following examples.*

i) Take the following non-anonymous refinement of the OG poverty ranking: $\mathsf{Y} \succcurlyeq_T^{g_1} \mathsf{Z}$ *iff* $\mathsf{Y} \succ_T^{g} \mathsf{Z}$ *or (*$\mathsf{Y} \sim_T^{g} \mathsf{Z}$, $Y_1 \nsupseteq T$ *and* $Z_1 \cap T \supseteq Y_1 \cap T$*). That ranking satisfies IIO, SMED and IBED but fails to satisfy AN.*

ii) Consider the following refinement of the OG poverty ranking: $\mathsf{Y} \succcurlyeq_T^{g^*} \mathsf{Z}$ *iff* $\mathsf{Y} \succ_T^{g} \mathsf{Z}$ *or (*$\mathsf{Y} \sim_T^{g} \mathsf{Z}$ *and* $\sum_{i \in N} \#(Y_i \smallsetminus T) \leq \sum_{i \in N} \#(Z_i \smallsetminus T)$*). That ranking satisfies AN, SMED and IBED but fails to satisfy IIO.*

iii) Consider again the universal indifference ranking: i.e. $\mathsf{Y} \succcurlyeq^{I} \mathsf{Z}$ *for any* $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$*. That preorder is a poverty ranking which does satisfy AN, IIO and IBED but violates SMED.*

iv) Consider the HC-refinement of the OG poverty ranking: $\mathsf{Y} \succcurlyeq_T^{g_h} \mathsf{Z}$ *iff* $\mathsf{Y} \succ_T^{g} \mathsf{Z}$ *or (*$\mathsf{Y} \sim_T^{g} \mathsf{Z}$ *and* $h_T(\mathsf{Y}) \geq h_T(\mathsf{Z})$*). That poverty ranking satisfies AN, IIO, SMED but violates IBED.*

5. Composite rankings

In this setion, we propose and characterize axiomatically two lexicographic rankings based on the HC and OG rankings and a third one based on a linear combination of the head-count and gap criteria.

A (HG)- *lexicographic poverty ranking* of opportunity profiles - under threshold T- is a binary relational system $\left(\mathcal{P}[X]^N, \succcurlyeq_T^{hg}\right)$ where $\succcurlyeq_T^{hg}$ is a preorder defined as follow: for any $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$, $\mathsf{Y} \succcurlyeq_T^{hg} \mathsf{Z}$ if and only if either $\mathsf{Y} \succ_T^{h} \mathsf{Z}$ or ($\mathsf{Y} \sim_T^{h} \mathsf{Z}$ and $g_T(\mathsf{Y}) \geq g_T(\mathsf{Z})$).

A (GH)- *lexicographic poverty ranking* of opportunity profiles - under threshold T- is a binary relational system $\left(\mathcal{P}[X]^N, \succcurlyeq_T^{gh}\right)$ where $\succcurlyeq_T^{gh}$ is a preorder defined as follow: for any $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$, $\mathsf{Y} \succcurlyeq_T^{gh} \mathsf{Z}$ if and only if either $\mathsf{Y} \succ_T^{g} \mathsf{Z}$ or ($\mathsf{Y} \sim_T^{g} \mathsf{Z}$ and $h_T(\mathsf{Y}) \geq h_T(\mathsf{Z})$).

A (HG)-*weighted poverty ranking* of opportunity profiles, under threshold T, is a binary relational system $\left(\mathcal{P}[X]^N, \succcurlyeq_T^{w}\right)$ where $\succcurlyeq_T^{w}$ is a preorder defined as follow: there exist $w_1, w_2 \in \mathbb{R}_{++}$ such that, for any $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$, $\mathsf{Y} \succcurlyeq_T^{w} \mathsf{Z}$ if and only if $w_1 h_T(\mathsf{Y}) + w_2 g_T(\mathsf{Y}) \geq w_1 h_T(\mathsf{Z}) + w_2 g_T(\mathsf{Z})$.

We now propose some axioms in order to characterize such rankings:

Axiom 7 (Qualified Independence of Balanced Essential Delations (Q-$IBED$)). *IFor any* $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$*, for any* $x, y, z \in X$ *and for any* $i \in N$*, such that* $Y_i \subset T$*,* $Z_i \subset T$*,* $y \in Y_i \cap T$ *and* $z \in Z_i \cap T$*:*

$$\mathsf{Y} \succcurlyeq_T \mathsf{Z} \quad \text{if and only if } (\mathsf{Y}_{-i}, Y_i \setminus \{y\}) \succcurlyeq_T (\mathsf{Z}_{-i}, Z_i \setminus \{z\}).$$

Axiom 8 (Conditional Dominance (CD)). *Let* $\succcurlyeq_T$ *be a poverty ranking with threshold* T*. Suppose there exist a positive integer* k *and* $f_1, ..., f_k \in \mathbb{R}^{\mathcal{P}[X]^N}$*, such that for all* $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$*,* $f_i(\mathsf{Y}) = f_i(\mathsf{Z})$*,* $i = 1, ..., k$ *entails* $\mathsf{Y} \sim_T \mathsf{Z}$*. Then, for all* $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$*,* $(f_1(\mathsf{Y}), ..., f_k(\mathsf{Y})) \neq (f_1(\mathsf{Z}), ..., f_k(\mathsf{Z}))$ *and* $f_i(\mathsf{Y}) \geq f_i(\mathsf{Z})$*,* $i = 1, ..., k$ *entails* $\mathsf{Y} \succ_T \mathsf{Z}$*.*

Axiom 9 (Non-Compensation (NC)). *Let $\succcurlyeq_T$ be a poverty ranking with threshold T. Suppose there exist a positive integer k and $f_1, ..., f_k \in \mathbf{R}^{\mathcal{P}[X]^N}$, such that:*

(i): for all $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$: if $f_i(\mathsf{Y}) = f_i(\mathsf{Z})$, $i = 1, ..., k$, then $\mathsf{Y} \sim_T \mathsf{Z}$,

(ii): there exist $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$ and $i^* \in \{1, ..., k\}$, such that $f_{i^*}(\mathsf{Y}) > f_{i^*}(\mathsf{Z})$ and $f_j(\mathsf{Z}) > f_j(\mathsf{y})$for any $j \in \{1, ..., k\}$, $j \neq i^*$.

Then for all $\mathsf{U}, \mathsf{V} \in \mathcal{P}[X]^N$: $\mathsf{U} \succ_T \mathsf{V}$ whenever $f_{i^*}(\mathsf{U}) > f_{i^*}(\mathsf{V})$.

Axiom 10 (Head-Count Priority (HP)). *Let $\succcurlyeq_T$ be a poverty ranking with threshold T, such that $\#T \geq 3$. For any $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$, if [there exist $i, j \in N$ and $x, y, z \in T$, with $x \neq y \neq z \neq x$, such that for any $l \neq i, j$, $Y_l = Z_l$, $Y_i = T \setminus \{x\}$, $Y_j = T \setminus \{y\}$, $Z_i = T$, and $Z_j = T \setminus \{x, y, z\}$],then $\mathsf{Y} \succ_T \mathsf{Z}$.*

Axiom 11 (Gap-Priority (GP)). *Let $\succcurlyeq_T$ be a poverty ranking with threshold T, such that $\#T \geq 3$. For any $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$, if [there exist $i, j \in N$ and $x, y, z \in T$, with $x \neq y \neq z \neq x$, such that for any $l \neq i, j$, $Y_l = Z_l$, $Y_i = T \setminus \{x\}$, $Y_j = T \setminus \{y\}$, $Z_i = T$, and $Z_j = T \setminus \{x, y, z\}$],then $\mathsf{Z} \succ_T \mathsf{Y}$.*

Axiom 12 (Non-Triviality of Indifference (NTI)). *Let $\succcurlyeq_T$ be a poverty ranking with threshold T. Suppose there exist a positive integer k and $f_1, ..., f_k \in \mathbf{R}^{\mathcal{P}[X]^N}$, such that for all $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$, $f_i(\mathsf{Y}) = f_i(\mathsf{Z})$, $i = 1, ..., k$, entails $\mathsf{Y} \sim_T \mathsf{Z}$. Then, there exist $\mathsf{U}, \mathsf{V} \in \mathcal{P}[X]^N$, $(f_1(\mathsf{U}), ..., f_k(\mathsf{U})) \neq (f_1(\mathsf{V}), ..., f_k(\mathsf{V}))$ and $\mathsf{U} \sim_T \mathsf{V}$.*

Axiom 13 (Cardinal Unit-Comparability (CUC)). *Let $\succcurlyeq_T$ be a poverty ranking with threshold T. Suppose there exist a positive integer k and $f_1, ..., f_k \in \mathbf{R}^{\mathcal{P}[X]^N}$, such that for all $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$: if $f_i(\mathsf{Y}) = f_i(\mathsf{Z})$, $i = 1, ..., k$ entails $\mathsf{Y} \sim_T \mathsf{Z}$. Posit*

$$\Phi = \left\{ \begin{array}{c} \varphi = (\varphi_1, ..., \varphi_k) : \varphi_i \in \mathbf{R}^{\mathbf{R}}, i : 1, ..., k \text{ such that there exist} \\ \alpha > 0, \beta_i \in \mathbf{R} \text{ with } \varphi_i(x) = \alpha x + \beta_i \text{ for any } x \in \mathbf{R} \end{array} \right\}.$$

Then, for all $\mathsf{Y}, \mathsf{Z}, \mathsf{V}, \mathsf{U} \in \mathcal{P}[X]^N$, $\mathsf{Y} \succcurlyeq_T \mathsf{Z}$ and $(f_1(\mathsf{U}), ..., f_k(\mathsf{U})) = ((\varphi_1 \circ f_1)(\mathsf{Y}), ..., (\varphi_k \circ f_k)(\mathsf{Y}))$ and $(f_1(\mathsf{V}), ..., f_k(\mathsf{V})) = ((\varphi_1 \circ f_1)(\mathsf{Z}), ..., (\varphi_k \circ f_k)(\mathsf{Z}))$ with $\varphi = (\varphi_1, ..., \varphi_k) \in \Phi$ entail $\mathsf{U} \succcurlyeq_T \mathsf{V}$.

The following Lemma is needed for our characterizations of composite rankings:

Lemma 1. *Let $\succcurlyeq_T$ be a poverty ranking on $\mathcal{P}[X]^N$ and a total preorder which satisfies AN, IIO, DEP and $Q-IBED$. Then, for any $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$, $(h_T(\mathsf{Y}), g_T(\mathsf{Y})) = (h_T(\mathsf{Z}), g_T(\mathsf{Z}))$ entails $\mathsf{Y} \sim_T \mathsf{Z}$.*

Proof. Let us suppose $h_T(\mathsf{Y}) = h_T(\mathsf{Z})$, $g_T(\mathsf{Y}) = g_T(\mathsf{Z})$. Also, notice that for any $\mathsf{U} \in \mathcal{P}[X]^N$, $h_T(\mathsf{U}) = h_T\left(\mathsf{U}_{|T}\right)$ and $g_T(\mathsf{U}) = g_T\left(\mathsf{U}_{|T}\right)$ by definition of h_T and g_T respectively. Therefore, $h_T\left(\mathsf{Y}_{|T}\right) = h_T\left(\mathsf{Z}_{|T}\right) = m$ and $g_T\left(\mathsf{Y}_{|T}\right) = g_T\left(\mathsf{Z}_{|T}\right) = k$ for some m, k non-negative (observe that $m = 0$ if and only if $k = 0$). Next, posit $\widehat{\mathsf{V}} = \left(\widehat{V}_i\right)_{i=1,...,n}$ with $\widehat{V}_i = T$ if $V_i \supseteq T$, and $\widehat{V}_i = \emptyset$ if

$V_i \cap T$ and note that $h_T\left(\mathsf{V}_{|T}\right) = h_T\left(\widehat{\mathsf{V}}\right)$ since $\widehat{\mathsf{V}}$ does not alter the set of poor population units in $\mathsf{V}_{|T}$. Next, $\mathsf{Y}_{|T} <_T \mathsf{Z}_{|T}$ if and only if $\widehat{\mathsf{Y}} <_T \widehat{\mathsf{Z}}$ by AN and a repeated application of $Q-IBED$ ($(m\,|T|-k)$ times). Moreover, since $h_T\left(\widehat{\mathsf{Y}}\right) = h_T\left(\widehat{\mathsf{Z}}\right)$ it follows by DEP that neither $\widehat{\mathsf{Y}} \succ_T \widehat{\mathsf{Z}}$ nor $\widehat{\mathsf{Z}} \succ_T \widehat{\mathsf{Y}}$. Therefore, $\widehat{\mathsf{Y}} \sim_T \widehat{\mathsf{Z}}$ because $<_T$ is a total preorder. Finally, $\mathsf{Y} \sim_T \mathsf{Y}_{|T}$ and $\mathsf{Z} \sim_T \mathsf{Z}_{|T}$ by repated applications of IIO. It follows, by transitivity, that $\mathsf{Y} \sim_T \mathsf{Z}$. ¤

Proposition 3. *Let $<_T$ bea poverty ranking of $\mathcal{P}[X]^N$ under threshold $T \subseteq X$, such that $\#T \geq 3$, and a total preorder. Then, $<_T = <_T^{hg}$ if and only if $<_T$ satisfies* AN*,* IIO*,* DEP, Q-IBED, CD, NC, *and* HP*.*

Proof. Let $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$, such that $\mathsf{Y} <_T^{hg} \mathsf{Z}$, then one of the following cases obtains:

a) $h_T(\mathsf{Y}) > h_T(\mathsf{Z})$ and $g_T(\mathsf{Y}) = g_T(\mathsf{Z})$
b) $h_T(\mathsf{Y}) > h_T(\mathsf{Z})$ and $g_T(\mathsf{Y}) > g_T(\mathsf{Z})$
c) $h_T(\mathsf{Y}) > h_T(\mathsf{Z})$ and $g_T(\mathsf{Z}) > g_T(\mathsf{Y})$
d) $h_T(\mathsf{Y}) = h_T(\mathsf{Z})$ and $g_T(\mathsf{Y}) > g_T(\mathsf{Z})$
e) $h_T(\mathsf{Y}) = h_T(\mathsf{Z})$ and $g_T(\mathsf{Y}) = g_T(\mathsf{Z})$

Under case a) b), d) $\mathsf{Y} \succ_T \mathsf{Z}$ by *CD.* Under case c), $\mathsf{Y} \succ_T \mathsf{Z}$ by Lemma 1 and *NC* and *HP*. In e) by Lemma 1 $\mathsf{Y} \sim_T \mathsf{Z}$. Hence, in any case, $\mathsf{Y} <_T \mathsf{Z}$.

Conversely, let $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$, such that $\mathsf{Y} <_T \mathsf{Z}$, then the following cases should be distinguished:

1) $h_T(\mathsf{Y}) > h_T(\mathsf{Z})$
2) $h_T(\mathsf{Z}) > h_T(\mathsf{Y})$
3) $h_T(\mathsf{Y}) = h_T(\mathsf{Z})$ and $g_T(\mathsf{Y}) > g_T(\mathsf{Z})$
4) $h_T(\mathsf{Y}) = h_T(\mathsf{Z})$ and $g_T(\mathsf{Z}) > g_T(\mathsf{Y})$
5) $h_T(\mathsf{Y}) = h_T(\mathsf{Z})$ and $g_T(\mathsf{Z}) = g_T(\mathsf{Y})$.

Under case 1), 3), $\mathsf{Y} \succ_T^{hg} \mathsf{Z}$ by definition. Under case 2), two subcases should be distinguished, namely either $g_T(\mathsf{Z}) \geq g_T(\mathsf{Y})$ or $g_T(\mathsf{Y}) > g_T(\mathsf{Z})$. If $g_T(\mathsf{Z}) \geq g_T(\mathsf{Y})$ then by CD $\mathsf{Z} \succ_T \mathsf{Y}$, a contradiction. If, on the contrary, $g_T(\mathsf{Y}) > g_T(\mathsf{Z})$ then, by Lemma 1 and NC and HP, $\mathsf{Z} \succ_T \mathsf{Y}$ a contradiction again. Moreover, under case 4) by CD $\mathsf{Z} \succ_T \mathsf{Y}$, a contradiction. Finally, under case 5), we have that $\mathsf{Y} \sim_T^{hg} \mathsf{Z}$ by definition. Hence, the desired result follows. ¤

Proposition 4. *Let $<_T$ be a poverty ranking of $\mathcal{P}[X]^N$ under threshold $T \subseteq X$ such that $\#T \geq 3$, and a total preorder. Then, $<_T = <_T^{gh}$ if and only if $<_T$ satisfies* AN*,* IIO*,* DEP, Q-IBED, CD, NC, *and* GP*.*

Proof. The proof replicates almost verbatim the previous one. We reproduce it here for the sake of completeness.

Let $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$, such that $\mathsf{Y} <_T^{gh} \mathsf{Z}$, then one of the following cases obtains:

a) $g_T(\mathsf{Y}) > g_T(\mathsf{Z})$ and $h_T(\mathsf{Y}) = h_T(\mathsf{Z})$

b) $g_T(\mathsf{Y}) > g_T(\mathsf{Z})$ and $h_T(\mathsf{Y}) > h_T(\mathsf{Z})$

c) $g_T(\mathsf{Y}) > g_T(\mathsf{Z})$ and $h_T(\mathsf{Z}) > h_T(\mathsf{Y})$

d) $g_T(\mathsf{Y}) = g_T(\mathsf{Z})$ and $h_T(\mathsf{Y}) > h_T(\mathsf{Z})$

e) $g_T(\mathsf{Y}) = g_T(\mathsf{Z})$ and $h_T(\mathsf{Y}) = h_T(\mathsf{Z})$

Under case a) b), d) $\mathsf{Y} \succ_T \mathsf{Z}$ by *CD.* Under case c), $\mathsf{Y} \succ_T \mathsf{Z}$ by Lemma 1 and *NC* and *HP*. In e) by Lemma 1 $\mathsf{Y} \sim_T \mathsf{Z}$. Hence, in any case, $\mathsf{Y} \succcurlyeq_T \mathsf{Z}$.

Conversely, let $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$, such that $\mathsf{Y} \succcurlyeq_T \mathsf{Z}$, then the following cases should be distinguished:

1) $g_T(\mathsf{Y}) > g_T(\mathsf{Z})$

2) $g_T(\mathsf{Z}) > g_T(\mathsf{Y})$

3) $g_T(\mathsf{Y}) = g_T(\mathsf{Z})$ and $h_T(\mathsf{Y}) > h_T(\mathsf{Z})$

4) $g_T(\mathsf{Y}) = g_T(\mathsf{Z})$ and $h_T(\mathsf{Z}) > h_T(\mathsf{Y})$

5) $g_T(\mathsf{Y}) = g_T(\mathsf{Z})$ and $h_T(\mathsf{Z}) = h_T(\mathsf{Y})$.

Under case 1), 3), $\mathsf{Y} \succ_T^{gh} \mathsf{Z}$ by definition. Under case 2), two subcases should be distinguished, namely either $h_T(\mathsf{Z}) \geq h_T(\mathsf{Y})$ or $h_T(\mathsf{Y}) > h_T(\mathsf{Z})$. If $h_T(\mathsf{Z}) \geq h_T(\mathsf{Y})$ then, by CD, $\mathsf{Z} \succ_T \mathsf{Y}$, a contradiction. If, on the contrary, $h_T(\mathsf{Y}) > h_T(\mathsf{Z})$ then, by Lemma 1 and NC and GP, $\mathsf{Z} \succ_T \mathsf{Y}$ a contradiction again. Moreover, under case 4), by CD, $\mathsf{Z} \succ_T \mathsf{Y}$ a contradiction. Finally, under case 5), we have that $\mathsf{Y} \sim_T^{gh} \mathsf{Z}$ by definition. Hence, the desired result. ¤

Proposition 5. *Let $\succcurlyeq_T$ be a poverty ranking of $\mathcal{P}[X]^N$ under threshold $T \subseteq X$ and a total preorder. Then, $\succcurlyeq_T = \succcurlyeq_T^w$ if and only if $\succcurlyeq_T$ satisfies* AN, IIO, DEP, Q-IBED, CD, NTI *and* CUC.

Proof. Checking that $\succcurlyeq_T^w$ is a poverty ranking which satisfies *AN, IIO, DEP, Q-IBED, CD, NTI* and *CUC* is straightforward. Then, we only need to prove the 'if' part. So, let $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$, such that $A = (h_T(\mathsf{Y}), g_T(\mathsf{Y})) \neq (h_T(\mathsf{Z}), g_T(\mathsf{Z})) = B$ and $\mathsf{Y} \sim_T \mathsf{Z}$ (such a pair exists by *NTI).* Next, observe that all points lying on line joining A and B are $\sim_T$ indifferent. Indeed, $A \sim_T B$ by hypothesis. Then, $A - A \sim_T B - A$, i.e. $O \sim_T B - A$ by CUC. Hence, for any $\lambda > 0$, $O \sim_T \lambda(B-A)$ by CUC , which, in turn, entails $A \sim_T \lambda(B-A)+A$. Similarly, $O \sim_T B-A$ implies that $-(B-A) \sim_T O$. Then, for any $\lambda > 0$, $\lambda(-(B-A) \sim_T O$ entails $A+\lambda(-(B-A)) \sim_T A$. Let us denote $w_1 x + w_2 y = k$, with $w_1, w_2 \in \mathbf{R}_+$ and $k \in \mathbf{R}$ the real line joining Y and Z. Moreover, observe that by CUC, $E = (h_T(\mathsf{Y}) + \delta_1, g_T(\mathsf{Y}) + \delta_2) \sim_T (h_T(\mathsf{Z}) + \delta_1, g_T(\mathsf{Z}) + \delta_2) = D$ for any $\delta_1, \delta_2 \in \mathbf{R}$. Therefore, all proper indifference curves are parallel to each other. Of course, there might exist a finite number of isolated points. But, then for each one of them, one can draw a line through it which is parallel to the other indifference curves. Finally, notice that by CD $\mathsf{U} \succ_T \mathsf{V}$ whenever $w_1 h_T(\mathsf{U}) + w_2 g_T(\mathsf{U}) = k_1$, $w_1 h_T(\mathsf{V}) + w_2 g_T(\mathsf{V}) = k_2$ and $k_1 > k_2$. ¤

6. Remarks

We have only considered comparisons of opportunity profiles for a fixed population. A possible extension of our analysis would be to compare the opportunties available to societies with different numbers of individuals. This would make it possible to rank opportunity profiles for different countries, different demographic groups, and for different time periods.