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COMPARISONS OF WELL-BEING: THEORY AND APPLICATIONS

WALTER BOSSERT and VALENTINO DARDANONI

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Comparisons of well-being: theory and applications*

Walter Bossert

Département de Sciences Économiques and CIREQ
Université de Montréal
C.P. 6128, succursale Centre-ville
Montréal QC H3C 3J7
Canada
walter.bossert@umontreal.ca

and

Valentino Dardanoni

Department of Economics
University of Palermo
Viale delle Scienze
90128 Palermo
Italy
vdardano@unipa.it

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1 Ranking rules

Let $K \in \mathbb{N}\setminus\{1\}$ and $N \in \mathbb{N}\setminus\{1,2\}$. The set $\{1,\ldots,K\}$ is a set of individual characteristics (such as levels of consumption, degree of participation in the community, quality of shelter) and the set $\{1,\ldots,N\}$ is a set of individuals. Let $\mathcal{C}$ be the set of all $N \times K$ matrices $C = (c_{ik})_{i\in\{1,\ldots,N\},k\in\{1,\ldots,K\}}$ with nonnegative entries. Furthermore, for all $C \in \mathcal{C}$ and for all $i \in \{1,\ldots,N\}$, let $c_i = (c_{i1},\ldots,c_{iK})$ be the i^{th} row of C. Our notation for vector inequalities is $\geq$, $>$, $\gg$. For $k \in \{1,\ldots,K\}$, e^k denotes the k^{th} unit vector in $\mathbb{R}^K_+$. $\mathbf{1}_K$ is the K-dimensional vector $(1,\ldots,1)$.

For a set S, $\mathcal{O}(S)$ is the set of all orderings on S. A *ranking rule* is a mapping $F\colon\mathcal{C} \to \mathcal{O}(\{1,\ldots,N\})$. For all $C \in \mathcal{C}$, $F(C)$ is the ranking, in terms of well-being, of the individuals with the characteristics vectors given by the rows of C. The following ranking rules are analyzed in this paper.

Single-characteristic rules. F is a single-charcateristic rule if and only if there exists $k \in \{1,\ldots,K\}$ such that, for all $C \in \mathcal{C}$ and for all $i,j \in \{1,\ldots,N\}$,

$$iF(C)j \Leftrightarrow c_{ik} \geq c_{jk}.$$

Perfect-substitute rules. F is a perfect-substitute rule if and only if there exists $\alpha \in \mathbb{R}^K_{++}$ such that, for all $C \in \mathcal{C}$ and for all $i,j \in \{1,\ldots,N\}$,

$$iF(C)j \Leftrightarrow \sum_{k=1}^{K} \alpha_k c_{ik} \geq \sum_{k=1}^{K} \alpha_k c_{jk}.$$

Perfect-complement rules. F is a perfect-complement rule if and only if there exists $\beta \in \mathbb{R}^K_{++}$ such that, for all $C \in \mathcal{C}$ and for all $i,j \in \{1,\ldots,N\}$,

$$iF(C)j \Leftrightarrow \min\{\beta_1 c_{i1},\ldots,\beta_K c_{iK}\} \geq \min\{\beta_1 c_{j1},\ldots,\beta_K c_{jK}\}.$$

Cobb-Douglas rules. F is a Cobb-Douglas rule if and only if there exists $\gamma \in \mathbb{R}^K_{++}$ such that, for all $C \in \mathcal{C}$ and for all $i,j \in \{1,\ldots,N\}$,

$$iF(C)j \Leftrightarrow \prod_{k=1}^{K} c_{ik}^{\gamma_k} \geq \prod_{k=1}^{K} c_{jk}^{\gamma_k}.$$

We now define two axioms that we assume to be satisfied throughout. They are analogous to the axioms generating welfarism in the standard social-choice framework. However, in our model, the interpretation is quite different: we compare individual well-being instead of alternatives, and the role played by the set N is analogous to that played by the set of alternatives in the classical social-choice model. Conversely, our characteristics play the role occupied by the individuals in the standard social-choice setting.

The first of the two axioms is our analogue of Pareto indifference. It requires that individuals with identical characteristics vectors are considered equally well-off.

Indifference at equality. For all $C \in \mathcal{C}$ and for all $i, j \in \{1, \ldots, N\}$, if $c_i = c_j$, then

$$iF(C)j \text{ and } jF(C)i.$$

Our second axiom is an independence condition which is the analogue of the axiom binary independence of irrelevant alternatives used in social-choice theory. It requires that the ranking of two individuals i and j depends on their characteristics vectors only.

Independence. For all $C, C' \in \mathcal{C}$ and for all $i, j \in \{1, \ldots, N\}$, if $c_i = c'_i$ and $c_j = c'_j$, then

$$iF(C)j \Leftrightarrow iF(C')j.$$

As in the social-choice literature, we obtain the analogue of the welfarism theorem (see d'Aspremont and Gevers, 1977, and Hammond, 1979): the two axioms introduced above, together with the definition of the domain of F, allow us to use a single ordering of characteristics vectors to compare any two individuals for any characteristics matrix. Again, note the analogy to the welfarism theorem in social choice, where a single ordering of utility vectors is sufficient to rank alternatives for any profile of utility functions, provided a social-welfare funcional (a mapping that assigns an ordering on the universal set of alternatives to every profile of utility functions in its domain) satisfies Pareto indifference and binary independence of irrelevant alternatives. The standard welfarism theorem employs an unlimted-domain assumption but its conclusion remains valid if all matrices with nonnegative entries (in the language of social-choice theory: all profiles of utility functions with range $\mathbb{R}$) are considered; see Bossert and Weymark (2002). Because we assume that there are at least three individuals and the domain of F is sufficiently rich, the proof of the following theorem can be obtained from the standard welfarism theorem by a reinterpretation of the variables involved; see, for example, Blackorby, Bossert and Donaldson (2002), Bossert and Weymark (2002) or d'Aspremont and Gevers (2002) for details.

Theorem 1 *F satisfies indifference at equality and independence if and only if there exists an ordering R on $\mathbb{R}^K_+$ such that, for all $C \in \mathcal{C}$ and for all $i, j \in \{1, \ldots, N\}$,*

$$iF(C)j \Leftrightarrow c_i R c_j.$$

We refrain from presenting a formal analogue to another standard theorem in social-choice theory stating that a property called strong neutrality is equivalent to the conjunction of Pareto indifference and binary independence of irrelevant alternatives (see, for example, Guha, 1972, Blau, 1976, d'Aspremont and Gevers, 1977, and Sen, 1977). As is straightforward to verify, given the domain assumption employed here, this result translates into our framework as well.

Given the conclusion of Theorem 1, we can express the above-defined rules in terms of the ordering R. We obtain the following formulations.

Single-characteristic orderings. R is a single-charcateristic ordering if and only if there exists $k \in \{1, \ldots, K\}$ such that, for all $x, x' \in \mathbb{R}_+^K$,

$$xRx' \Leftrightarrow x_k \geq x'_k.$$

Perfect-substitute orderings. R is a perfect-substitute ordering if and only if there exists $\alpha \in \mathbb{R}_{++}^K$ such that, for all $x, x' \in \mathbb{R}_+^K$,

$$xRx' \Leftrightarrow \sum_{k=1}^{K} \alpha_k x_k \geq \sum_{k=1}^{K} \alpha_k x_k.$$

Perfect-complement orderings. R is a perfect-complement ordering if and only if there exists $\beta \in \mathbb{R}_{++}^K$ such that, for all $x, x' \in \mathbb{R}_+^K$,

$$xRx' \Leftrightarrow \min\{\beta_1 x_1, \ldots, \beta_K x_K\} \geq \min\{\beta_1 x'_1, \ldots, \beta_K x'_K\}.$$

Cobb-Douglas orderings. R is a Cobb-Douglas ordering if and only if there exists $\gamma \in \mathbb{R}_{++}^K$ such that, for all $x, x' \in \mathbb{R}_+^K$,

$$xRx' \Leftrightarrow \prod_{k=1}^{K} x_k^{\gamma_k} \geq \prod_{k=1}^{K} x_k'^{\gamma_k}.$$

For future reference, we conclude this section with a definition of the positional orderings. For $x \in \mathbb{R}_+^K$, we let x^r be a rank-ordered permutation of x such that $x_1^r \leq x_2^r \leq \ldots \leq x_K^r$.

Positional orderings. R is a positional ordering if and only if there exists $k \in \{1, \ldots, K\}$ such that, for all $x, x' \in \mathbb{R}_+^K$,

$$xRx' \Leftrightarrow x_k^r \geq x_k'^r.$$

2 Additional axioms

Let P and I denote the asymmetric factor and the symmetric factor associated with the ordering R (called a social-welfare ordering in Gevers, 1979) of Theorem 1. Given the theorem, we define further axioms in terms of this ordering R rather than in terms of the function F in order to simplify notation. Equivalent conditions could be formulated for F.

We begin with a continuity axiom. It requires that 'small' changes in the characteristics vectors do not lead to 'large' changes in the ranking.

Continuity. For all $x \in \mathbb{R}^K_+$, the sets

$$\{x' \in \mathbb{R}^K_+ \mid xRx'\}$$

and

$$\{x' \in \mathbb{R}^K_+ \mid x'Rx\}$$

are closed.

Convexity requires the weak upper contour sets of R to be convex.

Convexity. For all $x \in \mathbb{R}^K_+$, the set

$$\{x' \in \mathbb{R}^K_+ \mid x'Rx\}$$

is convex.

The following dominance condition ensure that the ordering R responds appropriately to increases in all characteristics.

Dominance. For all $x, x' \in \mathbb{R}^N_+$, if $x \gg x'$, then

$$xPx'.$$

A condition that is related in spirit demands that an increase in one of the characetristics without a decrease in any of the others leads at least in some situations to an improvement according to R.

Sensitivity. For all $k \in \{1, \ldots, K\}$, there exist $x \in \mathbb{R}^K_+$ and $\xi \in \mathbb{R}_{++}$ such that

$$(x + \xi e^k)Px.$$

The next class of axioms is parametrized by a K-dimensional vector of positive coefficients. It requires all permutations of a weighted vector (where the weights are given by the parameters) to be indifferent to the original. For the formulation of this axiom, let $a \in \mathbb{R}^K_{++}$.

a-anonymity. For all $x \in \mathbb{R}^K_+$ and for all one-to-one mappings $\pi: \{1, \ldots, K\} \to \{1, \ldots, K\}$,

$$(a_{\pi(1)}x_{\pi(1)}, \ldots, a_{\pi(K)}x_{\pi(K)})I(a_1x_1, \ldots, a_Kx_K).$$

Clearly, $\mathbf{1}_K$-anonymity is the standard anonymity axiom.

We conclude with some axioms concerning invariance properties of R with respect to changes in the measurement scales used for the various characetristics. These properties are, of course, analogous to the information-invariance assumptions regarding the measurablity and the interpersonal comparability of individual utilities imposed on social-welfare functionals.

The first information-invariance property is independent interval-scale invariance. It assumes that all characteristics are measured by means of independent interval scales. The resulting requirement is that R is insensitive with respect to K-tuples of independent increasing affine transformations.

Independent interval-scale invariance. For all $x, x' \in \mathbb{R}^K_+$, for all $\lambda \in \mathbb{R}^K_{++}$ and for all $\delta \in \mathbb{R}^K_+$,

$$(\lambda_1x_1 + \delta_1, \ldots, \lambda_Kx_K + \delta_K)R(\lambda_1x'_1 + \delta_1, \ldots, \lambda_Kx'_K + \delta_K) \Leftrightarrow xRx'.$$

The next axiom is expressed in terms of a reference vector $a \in \mathbb{R}^K_{++}$. A common ordinal scale is applied to the weighted characteristic values, where the weights are given by the components of a.

a-proportional ordinal-scale invariance. For all $x, x' \in \mathbb{R}^K_+$ and for all increasing functions $\phi: \mathbb{R}_+ \to \mathbb{R}$,

$$(\phi(x_1)/a_1, \ldots, \phi(x_K)/a_K)R(\phi(x'_1)/a_1, \ldots, \phi(x'_K)/a_K) \Leftrightarrow xRx'.$$

Analogously to the interpretation of $\mathbf{1}_K$-anonymity as the usual anonymity axiom, $\mathbf{1}_K$-proportional ordinal-scale invariance is the standard invariance property with respect to common ordinal scales for all characteristics.

The following invariance requirement is based on the assumption that the characteristics levels can be measured in terms of independent translation scales.

Independent translation-scale invariance. For all $x, x' \in \mathbb{R}^K_+$ and for all $\delta \in \mathbb{R}^K_+$,

$$(x+\delta)R(x'+\delta) \Leftrightarrow xRx'.$$

Finally, we introduce an invariance requirement that applies if the characteristics are measured with independent ratio scales.

Independent ratio-scale invariance. For all $x, x' \in \mathbb{R}^K_+$ and for all $\lambda \in \mathbb{R}^K_{++}$,

$$(\lambda_1 x_1, \ldots, \lambda_K x_K)R(\lambda_1 x'_1, \ldots, \lambda_K x'_K) \Leftrightarrow xRx'.$$

3 Characterization results

We begin with a charcaterization of the single-characteristic orderings. It is obtained by adapting a continuous version of Sen's (1970) strengthening of Arrow's (1951, 1963) impossibility theorem to our framework. Sen (1970) showed that the conclusion of Arrow's theorem remains valid if Arrow's information-invariance assumption with repect to ordinal noncomparability is weakened to information invariance with respect to cardinal noncomparability (see also Bossert and Weymark (2002) for a discussion). In our setting, this result translates into the following theorem.

Theorem 2 *R satisfies continuity, dominance and independent interval-scale invariance if and only if R is a single-characteristic ordering.*

Next, we apply a result due to Roberts (1980) in order to characterize the class of perfect-substitute orderings. See also Blackwell and Girshick (1954) for a version of this result in the context of choice under uncertainty. A geometric proof can be found in Blackorby, Donaldson and Weymark (1984).

Theorem 3 *R satisfies continuity, dominance, sensitivity and independent translation-scale invariance if and only if R is a perfect-substitute ordering.*

Proof. Clearly, the perfect-substitute orderings satisfy the axioms in the theorem statement. Conversely, suppose R satisfies the required axioms. It is straightforward to verify that Roberts' (1980) characterization result for social-welfare orderings defined on $\mathbb{R}^K$ is true on $\mathbb{R}^K_+$ as well. Therefore, continuity, dominance and independent translation-scale invariance together imply that R ranks any two characteristics vectors by comparing the weighted sums of their components, where all weights are nonnegative and at least one weight is positive. Sensitivity now implies that all weights are positive, which implies that R is a perfect-substitute ordering. ■

The next characterization is more novel than the two previous ones in that it does not immediately follow from a corresponding result in social-choice theory. In particular, we utilize the parametrized anonymity and information-invariance axioms introduced above to axiomatize the perfect-complement orderings. Rather than the entire class of perfect-complement orderings, the orderings are characterized one paremeter vector at a time.

Theorem 4 *Let $a \in \mathbb{R}^K_{++}$. R satisfies continuity, convexity, dominance, a-anonymity and a-proportional ordinal-scale invariance if and only if R is the perfect-complement ordering with $\beta = a$.*

Proof. Let $a \in \mathbb{R}^K_{++}$. Again, it is immediate that the perfect-complement orderings satisfy the required axioms. To prove the converse implication, suppose that R satisfies the axioms. Define the ordering Q on $\mathbb{R}^K_+$ as follows. For all $x, x' \in \mathbb{R}^K_+$,

$$xQx' \Leftrightarrow (x_1/a_1, \ldots, x_K/a_K)R(x'_1/a_1, \ldots, x'_K/a_K).$$

Because of the properties of R, Q satisfies continuity, dominance, $\mathbf{1}_K$-anonymity and $\mathbf{1}_K$-proportional ordinal-scale invariance. Using a version of results by Gevers (1979) and Roberts (1980) that applies to $\mathbb{R}^K_+$, it follows that Q is a positional ordering. Furthermore, because R is convex, so is Q. The only convex positional ordering is the perfect-complement ordering with $\beta = \mathbf{1}_K$. Using the definition of Q, we have

$$xRx' \Leftrightarrow (a_1x_1, \ldots, a_Kx_K)Q(a_1x'_1, \ldots, a_Kx'_K)$$

for all $x, x' \in \mathbb{R}^K_+$ and, substituting the perfect-complement ordering with $\beta = \mathbf{1}_K$ for Q, it follows that R is the perfect-complement ordering with $\beta = a$. ■

We conclude this section with a charcaterization of the class of Cobb-Douglas orderings which is based on an axiomatization due to Tsui and Weymark (1997).

Theorem 5 *R satisfies continuity, dominance, sensitivity and independent ratio-scale invariance if and only if R is a Cobb-Douglas ordering.*

Proof. Clearly, the Cobb-Douglas orderings satisfy the required axioms. Conversely, suppose R has all the properties listed in the theorem statement. Because R satisfies continuity, dominance and independent ratio-scale invariance, Theorem 5 of Tsui and

Weymark (1997) can be invoked to establish that R must belong to the generalization of the class of Cobb-Douglas orderings such that all components of the parameter vector γ are nonnegative and at least one is positive. Sensitivity implies that γ must be a vector with positive components only, and we obtain the Cobb-Douglas orderings. ■

That the axioms used in each of the above characterization results are independent is easy to see.

4 Applications

5 Concluding remarks

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