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Taxing capital income as Pigouvian correction: the role of discounting future

Valeria De Bonis* Luca Spataro†

Dipartimento di Scienze Economiche, Università di Pisa,

via C. Ridolfi 10, 56124, Pisa, Italy

Abstract

In this work we take up the issue of optimal dynamic capital income taxation in an infinitely lived representative agent framework. We find that, besides the traditional arguments descending from the shape of the utility function, the zero capital tax result does not generally hold when the government discount rate differs from the individual one. As intuitive Pigouvian considerations would suggest, along the transition path, even with separable and homothetic utility functions, capital income is taxed (subsidized) when the government is less (more) impatient than individuals are. Instead, we obtain a counterintuitive asymmetry as for the steady state, since in the long run only the case for subsidizing capital is confirmed. This is because, when the government discounts the future more heavily than private agents do, the explosive distortionary effect of taxation impedes to hit capital income. By the same reasoning, the asymmetry disappears in the special case of logarithmic utility function, since the anticipated policy path does not affect current individual choices and, thus, the cumulative distortionary effect of taxes is ruled out.

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*Tel.: +39 050 2216382, fax: +39 050 598040, e-mail: debonis@ec.unipi.it.

†Tel.: +39 050 2216333, fax: +39 050 598040, e-mail: l.spataro@ec.unipi.it.

1 Introduction

In this paper we reconsider the problem of dynamic optimal taxation. Such topic, dating back to the zero capital tax result obtained by the pioneering works of Judd (1985) and Chamley (1986), has been increasingly debated over the last decades. However, only recently several studies have shed light on the strict similarity between the optimal taxation problem in a dynamic set up and the more traditional, static problem of commodity taxation. In particular, the work by Atkeson and al. (1999) has highlighted such a relationship by adopting what Atkinson and Stiglitz (1980) call the "primal approach" to the Ramsey problem of taxation. The point, however, is still controversial: in fact, on the one hand, Atkeson and al. (1999) demonstrate the validity of the zero tax result in a framework with infinitely-lived individuals and under different hypotheses concerning agents' heterogeneity, growth and openness of the economy[1]; on the other hand, Erosa and Gervais (2002), by undertaking the same approach, point out that the Chamley-Judd result does not generally apply, neither at the steady state nor along the transition path in overlapping generations models with a life-cycle.

As shown by Judd (1999), this contrast can be explained considering that a tax on capital income is equivalent to a tax on future consumption. In infinitely-lived, representative agent models, the economy converges to a long run allocation in which individual consumption is constant; this implies that the elasticity of consumption is also constant: hence, since uniform taxation of present and future consumption applies, the tax rate on capital income must be zero. In overlapping generations-life cycle models, instead (Atkinson and Sandmo (1980), Escolano (1991), Renström (1999), Erosa and Gervais (2002)), since the individual time path of consumption and leisure is almost never constant through life, the government will find it optimal to tax or

[1] Note that the zero tax rate result applies also to human capital (Jones, Manuelli and Rossi (1997)). Judd (1999) explains it in terms of the difference between taxation of stocks and of flows. One can note that in these models, where individuals are identical as for wealth holdings and consumption/leisure choices and the interest rate equals the subjective discount rate (so that the flow does not constitute new wealth in the sense of increasing utility, but is only the minimum required remuneration for postponing consumption), taxing capital income effectively means only discriminating against savings (future consumption).

subsidize interest income at different rates over the life cycle[2].

In fact the work by Judd (1999) has provided a dramatic contribution in clarifying both the economic nature of the zero capital income tax result and the circumstances under which such result can be violated. His findings can be posed in the following way: when the utility function is separable and not homothetic, the inverse elasticity result applies; in fact, since taxing capital income today is equivalent to taxing tomorrow consumption, the government will find it optimal to levy (increase) such tax today if consumption is getting more "inelastic" tomorrow. On the other hand, if leisure and consumption are intratemporally not separable, he shows that there is room for taxing capital income in the short run. However, Judd does not provide an economic explanation of this result[3].

The mentioned works are most related with the present one, in that we tackle the Ramsey problem under the primal approach. However, we depart from Atkeson and al. (1999) and Erosa and Gervais (2002) in that we tackle this issue in a continuous time framework with a representative individual. Judd (1999) addresses the issue in continuous time, but does not adopt the primal approach.

More importantly, we allow the government and the individual rates of time preference to differ and discuss the consequences of this hypothesis. In fact, although equality between them is normally assumed in representative agent models[4], this can be not necessarily true in real cases. First of all, the public choice literature has delivered examples in which the government can

[2]In particular, Erosa and Gervais (2002) show that, since (non taxable) leisure varies over time, there are two roles for interest taxation: 1) in the lack of age dependent wage tax rates, a positive interest tax makes leisure more expensive as individuals get old; 2) if consumption and leisure are complements, consumption should be taxed more when it is higher, i.e. in the old age, in line with the Corlett-Hague rule.

[3]Notice that this result contrasts sharply with the assertion by Chamley (1986) that the capital tax will differ from zero (and, typically, be at confiscatory levels) only at the very first period of the policy introduction (or, in continuous time, for a limited period). The difference is also in the forces driving the two results: in Judd (1999), the variation of the elasticity of the marginal utility of consumption over time; in Chamley (1986), the trade-off between the benefit from lump-sum taxation of existing asset holdings (in terms of future lower distortionary taxation) and the costs due to individual reactions.

[4]A notable exception is the work by Arrow and Kurz (1970), who however focus on the case of constant propensity to save.

be more myopic than individuals due to electoral cycles; on the other hand, paternalistic arguments have been claimed for implementing public social security systems whenever private agents do not properly take the future into consideration; finally, OLG frameworks have been proposed in which the government intertemporal discount rate turns out to differ systematically from the individuals'one: typically, this happens when the economy is characterized by a disconnection among generations, due to the absence of altruism (or limited intra-family altruism[5]). Although these more sophisticated models are perhaps the most natural scenarios for exploring the difference between the public and private discount rates[6], analyzing it in an ILRA framework enables us to isolate the effects of such factor on the capital income tax and to provide a general, intuitive explanation of the results in terms of the traditonal optimal taxation theory.

The main contributions of our paper are the following: first, we develop the conditions under which the solution of the Ramsey problem can be decentralized as a competitive equilibrium in a continuous time framework.

Second, by following the primal approach we can reinterpret the Judd (1999) result in terms of the so called "general equilibrium elasticity" of consumption. In particular, if separability between consumption and leisure is assumed, we formally derive the expression for the capital income tax rate in terms of price elasticity of consumption. On the other hand, in the case of non separability between consumption and labor, we provide an interpretation of the result in terms of the Corlett-Hague rule.

Third, we highlight the role of the difference between the individual and the government discount rates as a determinant of the capital income tax rate. In this respect, our main finding is the following: when the government is more (less) impatient than individuals, it finds it optimal to raise positive (negative) taxes on capital income, that is, to penalize (subsidize) future consumption. However, at the steady state, we end up with a counterintuitive asymmetry: in fact, while the case for subsidizing is still valid, there is no room for taxing capital. The economic justification for such somehow striking outcome is that the deadweight loss brought about by the distortionary tax on capital income behaves in a completely different way according to the

[5]See, for example, Blanchard (1985), Buiter (1988) and Weil (1989).

[6]For applications to OLG models, see De Bonis and Spataro (2003a) and (2003b).

sign of the difference between the time discount rates: more precisely, in the long run, while tending to zero when the government is more impatient, such distortion explodes to infinity in the opposite case, thus offsetting the welfare gain produced by taxation. Consequently, in this case the optimal tax rate on capital income should be zero.

By the same reasoning, however, it is possible to show that, when the utility function is separable and logarithmic (that is, elasticity of intertemporal substitution of consumption equal to unity), the announced policy affects current consumption only to the extent of the change of the current interest rate. Since agents' decisions do not depend on the anticipated path of the capital income tax rates, which eliminates the (explosive) cumulative distortionary burden of the policy, the symmetry in the results is restored: it is optimal for the policymaker to tax or subsidize capital accumulation in the long run according to whether it is more or less impatient than individuals[7].

At the end of these introductory remarks, we must mention that the application of the Ramsey approach to a dynamic framework raises a time inconsistency problem. We do not tackle such third best issues, but adopt the usual assumption that the government has access to a commitment technology.

The paper proceeds as follows: in the first section we lay out our framework; next, we present the Ramsey problem according to the primal approach and derive the solution; after discussing the results, we propose some examples to clarify the role of the forces behind them. Concluding remarks and a technical appendix end the work.

2 The model

Let us analyze a neoclassical-production-closed economy in which a large number of agents and firms operate.

Only one production-consumption good and labor do exist in this economy and, for the sake of simplicity, we abstract from growth. Private agents, who are infinitely lived and identical, are all born at the beginning of the

[7] We should note that this line of reasoning is equivalent to that presented by Lansing (1999). Here, however, we end up with an opposite conclusion (Lansing's long run zero tax result would be restored, again, if public and private preferences were the same).

economy, receive a windfall amount of per-capita wealth $a(0) = \overline{a}$; furthermore, they offer labor and capital services to firms by taking the net-of-tax factor prices, $\widetilde{w}(t)$ and $\widetilde{r}(t)$ as given. Each firm i owns a constant return to scale technology $F(k^i(t), l^i(t))$ satisfying the Inada conditions and which transforms the factors into production-consumption units. Finally, a set of policies is available to the government whereby it is able to finance an exogenous and constant stream of public expenditure g, by issuing internal debt $B(t)$ and raising proportional taxes both on capital and on labor services (and referred to as $\tau^k(t)$ and $\tau^l(t)$)[8].

2.1 Private agents

The agents' preferences can be represented by the following instantaneous utility function:

$$U(c(t), l(t))$$

where $c(t)$ and $l(t)$ are instantaneous consumption and labor supply respectively. Such a utility function is strictly increasing in consumption and decreasing in labor, strictly concave, and satisfies the standard Inada conditions.

As usual, the aim of such agents is to maximize the discounted sum of the stream of lifetime utils by choosing the optimal time path of consumption (savings) and labor hours under the budget constraint.

That is:

$$\max_{\{c(t),l(t)\}_0^\infty} \int_0^\infty e^{-\beta t} U(c(t), l(t))\, dt \tag{1}$$

$$sub\ \dot{a}(t) = \widetilde{r}(t)\, a(t) + \widetilde{w}(t)\, l(t) - c(t) \tag{2}$$

$$\lim_{t\to\infty} a(t)\, e^{-\int_0^t \widetilde{r}(v)dv} = 0, \quad a(0) = \overline{a}$$

where β is the intertemporal discount rate, a the agent's wealth; the notation $\dot{()}$ indicates the derivative with respect to time, while $\widetilde{r}(t) = r(t)\left(1 - \tau^k(t)\right)$

[8]We do not consider consumption taxes since they could be used in association with labor income taxes to mimic a lump sum levy on initial asset holdings (see Chari and Kehoe (1999)).

and $\widetilde{w}(t) = w(t)\left(1 - \tau^l(t)\right)$ are the net-of-tax factor prices. The first order conditions of this problem imply[9]:

$$U_c = p \tag{3}$$

$$U_l = -p\widetilde{w} \tag{4}$$

$$-\tilde{r}p = \dot{p} - \beta p \tag{5}$$

where the expression U_i is the partial derivative of the utility function with respect to *time* t argument $i = c, l$ and p is the current value shadow price of wealth. According to such conditions, it can be shown that the growth rates of consumption and labor are:

$$\frac{\dot{c}}{c} = (\widetilde{r} - \beta)\frac{1}{\theta_c} - \frac{\theta_{cl}}{\theta_c}\frac{\dot{l}}{l} \tag{6}$$

$$\frac{\dot{l}}{l} = \frac{1}{\theta_l}\frac{\left[(\widetilde{r} - \beta)\left(1 - \frac{\theta_{lc}}{\theta_c}\right) - \frac{\dot{\widetilde{w}}}{\widetilde{w}}\right]}{1 - \frac{\theta_{cl}\theta_{lc}}{\theta_c\theta_l}} \tag{7}$$

with $\theta_j = -\frac{U_{jj}j}{U_j}$, $j = c, l$, the elasticity of the marginal utility and $\theta_{ij} = -\frac{U_{ij}j}{U_i}$. Notice that at the steady state one would expect both growth rates (and $\frac{\dot{\widetilde{w}}}{w}$ as well) to be constant, and, in particular in such a model with no growth, equal to zero. Finally, in case the utility function is additively separable in consumption and labor, the growth rates above are: $\frac{\dot{c}}{c} = (\widetilde{r} - \beta)\frac{1}{\theta_c}$ and $\frac{\dot{l}}{l} = (\widetilde{r} - \beta)\frac{1}{\theta_l}$.

2.2 Firms

As mentioned, firms run their business in a perfectly competitive framework without uncertainty and investment adjustment costs. As a consequence, in each instant firms hire capital and labor services according to their market prices (gross of taxes) and in order to maximize their current period profits. This means that, for each firm i and each instant t:

[9] We omit arguments of the functions whenever such arguments are clear from the context.

$$F_{K^i} \equiv \frac{dF\left(K^i, L^i\right))}{dK^i} = r \tag{8}$$

$$F_{L^i} \equiv \frac{dF\left(K^i, L^i\right)}{dL^i} = w. \tag{9}$$

Notice that due to the CRS assumption, the Euler's theorem applies and the conditions above can be expressed for the economy as a whole, in per capita terms.

$$f_k = r \tag{8'}$$

$$f_l = w. \tag{9'}$$

2.3 Government and market clearing conditions

The government fixes an amount of exogenous public expenditure and finances it through proportional taxes on income and by issuing debt. There is no constraint on the amount of debt (neither on the levels nor on the growth rates)[10].

Thus, one obtains the usual condition in per capita terms:

$$\dot{b} = rb + g - \tau^k ra\text{-}\tau^l wl. \tag{10}$$

Finally, since the market clearing condition implies that, at each date, the sum of capital and debt equal aggregate private wealth, that is:

$$a = k + b \tag{11}$$

then equation (10) can be also written as:

$$\dot{b} = \widetilde{r}b + g - \tau^k rk\text{-}\tau^l wl. \tag{10'}$$

[10] The only constraint on the debt law of motion is the usual no-Ponzi game condition, namely: $\lim_{t\to\infty} b\left(t\right) e^{-\int_0^t \widetilde{r}(v)dv} = 0$, and the starting condition $b\left(0\right) = \bar{b}$.

3 The Ramsey problem

The Ramsey problem consists in the maximization of a social welfare function through the choice of taxes, subject to the constraints that the resulting allocation be a (feasible) competitive equilibrium (with distortionary taxation) and that the required amount of revenue be raised. The problem has two formulations (Atkinson and Stiglitz (1980)): the "dual" approach, that exploits the properties of the indirect utility function and takes dual prices and tax rates as control variables; and the "primal" approach, characterized by the maximization of a direct utility function through the choice of quantities. Under the primal approach, since the optimal policy problem consists in the choice of an allocation, the central point is restricting the set of allocations among which the government can choose to those that can be decentralized as a competitive equilibrium[11]. In this paragraph we define a competitive equilibrium and the constraints that must be imposed to the policymaker problem, in order to achieve such a competitive outcome (so called "implementability" and "feasibility" constraints).

Definition 1 *A competitive equilibrium is: a) an infinite sequence of policies* $\pi = \left\{\tau^k(t), \tau^l(t), b(t)\right\}_0^\infty$, *b) allocations* $\{c(t), l(t), k(t)\}_0^\infty$ *and c) prices* $\{w(t), r(t)\}_0^\infty$ *such that, at each instant* t : *b) satisfies equation* (1) *subject to* (2)*, given a) and c); c) satisfies equation* (8′) *and equation* (9′)*; equations* (14) *and* (10′) are satisfied.

Such allocations are often referred to as "implementable".

As for the implementability constraint, it can be obtained via the following steps: first, by taking equation (2) and multiplying both sides by $e^{-\int_0^t \widetilde{r}(v)dv}$, we can write the following expression:

$$\frac{d\left[a(t)\, e^{-\int_0^t \widetilde{r}(v)dv}\right]}{dt} = e^{-\int_0^t \widetilde{r}(v)dv}\left[\widetilde{w}(t)\, l(t) - c(t)\right];$$

next, multiplying both sides by $p(t)$ and exploiting the individuals' FOC (3 to 5) we obtain:

[11] To clarify this point, let us consider that at a certain date t a given policy is announced: then, consumers and firms will react accordingly maximizing their objective functions, taking the new prices as given. As a consequence, via the adjustment process, the policy will generate a competitive equilibrium allocation.

$$p(0)\, e^{-\int_0^t((\widetilde{r}(v)-\beta))dv} \frac{d\left[a(t)\, e^{-\int_0^t \widetilde{r}(v)dv}\right]}{dt} = -e^{-\int_0^t \widetilde{r}(v)dv} \left[U_l(t)\, l(t) + U_c(t)\, c(t)\right]$$

$$\Rightarrow -U_c(0) \frac{d\left[a(t)\, e^{-\int_0^t \widetilde{r}(v)dv}\right]}{dt} = e^{-\beta t} \left[U_l(t)\, l(t) + U_c(t)\, c(t)\right]$$

finally, by integrating out and exploiting the individual's transversality condition, it follows:

$$\int_0^\infty e^{-\beta t} \left[U_{c(t)} c(t) + U_{l(t)} l(t)\right] dt = a(0)\, U_{c(0)} \tag{12}$$

which is referred to as the "implementability constraint".

Finally, we define the feasibility constraint. Writing expression (2) in the following way:

$$\dot{a}(t) = r(t)\, a(t) + w(t)\, l(t) - c(t) - \tau^k(t)\, r(t)a(t)\text{-}\tau^l(t)\, w(t)l(t)$$

exploiting the market clearing condition and the government budget constraint, we get:

$$\dot{a}(t) - \dot{b}(t) = r(t)\, k(t) + w(t)\, l(t) - c(t) - g. \tag{13}$$

Finally, since the economy has a CRS production technology, we get the "feasibility constraint":

$$\dot{k}(t) = f(k(t), l(t)) - c(t) - g. \tag{14}$$

This expression states that the amount of capital in the economy grows at each instant t thanks to the private savings, net of public spending.

In the light of the definitions given above, the following proposition holds:

Proposition 1 *An allocation is a competitive equilibrium if and only if it satisfies implementability and feasibility.*

Proof. The first part of the proposition is true by construction. The reverse (any allocation satisfying implementability and feasibility is a competitive equilibrium) is provided in Appendix 1. ■

Concluding, a final issue is worth being recalled, since it potentially restricts the set of policies and allocations which are effectively implementable by the policymaker. This point concerns the "time inconsistency" problem affecting optimal taxation when a dynamic set up is considered: as several authors have pointed out[12], typically the government has incentives to deviate from the announced (ex-ante) second best policy, upon achieving the instant in which the policy is phased in; in fact this happens because one of the factors to be taxed (i.e. capital) ex-post is perfectly rigid and now should be taxed more heavily. But (perfectly rational) individuals take into account such incentives and react to them accordingly: as a result, a different, suboptimal (what is referred to as the "third best") equilibrium is achieved. In our work we do not analyze such issue and suppose that the policy maker has a "commitment" technology so that it can tie itself to the second best policy path[13].

3.1 The primal approach solution

As already mentioned, according to the primal approach to the Ramsey problem the policy maker chooses quantities (allocations) so as to maximize a social welfare function, under the constraints that such allocations implement a competitive equilibrium. Let us now define the Ramsey problem and the solution.

$$\max_{\{c(t),l(t),k(t)\}_0^\infty} \int_0^\infty e^{-\gamma t} U(c,l)\, dt$$

$$\textit{subject to} \int_0^\infty e^{-\beta t} \left[U_c c + U_l l\right] dt = a(0)\, U_{c(0)}$$

$$\textit{and } \dot{k} = f(k,l) - c - g.$$

[12] For an analysis of such issue see, for example, Renström (1999).

[13] Such a hypothesis implies also that the capital tax at the beginning of the policy (which we suppose to be introduced at the beginning of time) is given, that is fixed exogenously at a level belonging to the $(0,1)$ interval. Should this not be the case, the government would have an incentive to confiscate the whole capital income since it would exactly be equivalent to a lump sum taxation. See Appendix C for a discussion of the starting level of capital tax.

$$a(0) = \overline{a}, \qquad \lim_{t\to\infty} ke^{-\int_0^t r(v)dv} = 0,$$

where γ is the government discount rate (which a priori is supposed not to necessarily coincide with the individual one, β).

This is an "isoperimetric problem", whose Hamiltonian can be written as follows[14]:

$$\overline{J}\left(t, \hat{\lambda}, \hat{\eta}, c, l, k\right) = e^{-\gamma t} U(c, l) + \eta \left(f(k, l) - c - g\right) \\ + \lambda e^{-\beta t} \left[U_c c + U_l l\right].$$

By expressing the Hamiltonian in current value terms

$$J\left(t, \hat{\lambda}, \hat{\eta}, c, l, k\right) = U(c, l) + \hat{\eta} \left(f(k, l) - c - g\right) \\ + \hat{\lambda} \left[U_c c + U_l l\right]$$

where $\hat{\lambda}$ is the current value multiplier associated to the implementability constraint, defined as $\hat{\lambda} = \lambda e^{(\gamma - \beta)t}$ and $\hat{\eta}$ is the current value multiplier associated to the feasibility constraint, the necessary conditions for the solutions are[15]:

$$U_c + \hat{\lambda} \left[U_{cc} c + U_c + U_{lc} l\right] = \hat{\eta} \tag{15}$$

$$U_l + \hat{\lambda} \left[U_{ll} l + U_l + U_{cl} c\right] = -\hat{\eta} f_l \tag{16}$$

$$-f_k \hat{\eta} = \dot{\hat{\eta}} - \gamma \hat{\eta} \tag{17}$$

[14] See Appendix 2 for the solution conditions of this problem. Notice that the Inada's conditions guarantee interiority of the solution. However, the first-order conditions for the Ramsey problem are necessary but not sufficient, due to the possibility of nonconvexity of the implementability constraint. Finally, the government budget constraint can be omitted by Walras's law, if the implementability and the feasibility constraints are satisfied.

[15] The exact formulation of the transversality condition is controversial. The one we have chosen has been shown to always apply (see Barro and Sala-i-Martin (1999), p.504. However, also the usual one $\lim_{t\to\infty} k(t)\, \eta(t)\, e^{-\int_0^t r(v)dv}$ could be used.

$$\dot{\hat{\lambda}} = (\gamma - \beta)\,\hat{\lambda} \tag{18}$$

$$\lim_{t\to\infty} \overline{J} = 0. \tag{19}$$

Now, by focusing on the capital income tax, differentiating equation (15) with respect to time, we get:

$$\left[U_{cc}\dot{c} + U_{cl}\dot{l}\right]\left[1 + \hat{\lambda} + \hat{\lambda}H_c\right] + \dot{\hat{\lambda}}U_c\,(1 + H_c) + \hat{\lambda}\dot{H}_c U_c = \dot{\hat{\eta}} \tag{20}$$

by substituting the expression of $\dot{\hat{\eta}}$ and $\dot{\hat{\lambda}}$ from equation (17) and (18) and reckoning that $\hat{\eta} = U_c\left(1 + \hat{\lambda}\,(1 + H_c)\right)$, we can write:

$$\frac{\dot{c}}{c} = \frac{U_c}{U_{cc}c}\left[(\gamma - f_k) + (\beta - \gamma)\,\frac{\hat{\lambda}\,(1 + H_c)}{\left[1 + \hat{\lambda}\,(1 + H_c)\right]} - \frac{\hat{\lambda}\dot{H}_c}{\left[1 + \hat{\lambda}\,(1 + H_c)\right]} + \theta_{cl}\frac{\dot{l}}{l}\right]$$

which can be written also as:

$$\frac{\dot{c}}{c} = \frac{1}{\theta_c}\left[(f_k - \gamma) - (\beta - \gamma)\,\frac{\hat{\lambda}\,(1 + H_c)}{\left[1 + \hat{\lambda}\,(1 + H_c)\right]} + \frac{\hat{\lambda}\dot{H}_c}{\left[1 + \hat{\lambda}\,(1 + H_c)\right]} - \theta_{cl}\frac{\dot{l}}{l}\right].$$

Now, substituting for the growth rate of consumption stemming from the individual optimization condition (equation 6), we get:

$$(\widetilde{r} - \beta) - \theta_{cl}\frac{\dot{l}}{l} = \left[(f_k - \gamma) - (\beta - \gamma)\,\frac{\hat{\lambda}\,(1 + H_c)}{\left[1 + \hat{\lambda}\,(1 + H_c)\right]} + \frac{\hat{\lambda}\dot{H}_c}{\left[1 + \hat{\lambda}\,(1 + H_c)\right]} - \theta_{cl}\frac{\dot{l}}{l}\right]$$

next, exploiting the equilibrium condition on the capital market it follows:

$$\tau^k = -\frac{1}{f_k}\left\{(\beta - \gamma)\left[\frac{1}{1 + \hat{\lambda}\,(1 + H_c)}\right] + \frac{\hat{\lambda}\dot{H}_c}{\left[1 + \hat{\lambda}\,(1 + H_c)\right]}\right\}. \tag{21}$$

3.2 Discussion of the results

We now characterize the results of the previous section by analyzing both the transition path and the steady state properties of the economy. As it will be made clear in the following discussion, a crucial element driving our results is, among others, the relationship between the government and the individual rate of time preference. In fact, by allowing such parameters to differ from each other, we can enrich the analysis of the factors determining both capital and income tax rates. More precisely, we show that the steady state zero capital tax rate result applies in general only if $\beta \leq \gamma$, while capital should be subsidized if $\beta > \gamma$.

Preliminarily, it is worth noting that equation (21) does not yield an explicit formula for τ^k, since H_c can depend upon the tax rates themselves. Moreover, we do not have any condition ensuring that the tax rate will be in the $(0, 1)$ interval, while we would suspect capital income taxes especially to get sticking at the upper boundary for a (finite) period of time since the introduction of the policy. However, in the rest of the work we maintain the assumption of interiority of the equilibrium tax rates, for $t > 0$.

Now, as for the capital income tax, the following proposition holds:

Proposition 2 *If the economy converges to a steady state, along the transition path, for $t > 0$, the tax on capital income is in general different from zero unless $\dot{H}_{c(t)} = 0$ and $\beta = \gamma$. At the steady state, the capital income tax is different from zero unless: i) $\beta \leq \gamma$ or ii) $H_c = -1$ and $\beta \neq \gamma$.*

Proof. The proof concerning the transition path is obvious. As for the steady state, since $\dot{H}_c = 0$, when $\beta = \gamma$, $\tau^k = 0$. The same result holds if $\beta < \gamma$, since $\hat{\lambda}$ tends to ∞ and $\frac{1}{1+\hat{\lambda}(1+H_c)}$ tends to 0. Instead, if $\beta > \gamma$, $\hat{\lambda}$ tends to 0 and τ^k is equal to $-\frac{(\beta-\gamma)}{f_k}$. Finally, if $H_c = -1$, at the steady state τ^k is equal to $-\frac{(\beta-\gamma)}{f_k}$, which is positive (negative) if $\beta < (>)\, \gamma$. ■

From the proposition above, it emerges that there are two crucial forces driving the time path of τ^k. The first one is the dynamics of $H_{c(t)}$ and the second one is the difference between β and γ. While the former replicates the results found in Judd (1999), the latter is a further source of taxation, which, although usually disregarded, can play a crucial role in both the transition and the steady state.

As it will be clear in the examples which follow, this new factor can be interpreted as a Pigouvian corrective element, which, however, has to be compared with the other side of the coin: the cost of the distortion it brings about. Before focusing on such outcome, we present a general re-interpretation of the already known results in terms of the more traditional, static optimal taxation analysis.

In order to explain the results let us call the first element the "General Equilibrium Elasticity Factor" (GEEF) and the second one the "Intertemporal Discount Factor" (IDF), so that the expression for τ^k can be written as:

$$\tau^k = -\frac{1}{f_k}\left\{GEEF + IDF\right\}.$$

In the analysis that follows we disentangle the role of each factor.

3.2.1 The role of the utility function

If we assume that $\beta = \gamma$, we can write the following corollary:

Corollary 1 *If $\beta = \gamma$, then the only source of capital income taxation in the short run is given by GEEF.*

To make this point clear, note that H_c is equal to -$(\theta_c + \theta_{cl})$, so that $\dot{H}_c =$ -$\left(\dot{\theta}_c + \dot{\theta}_{cl}\right)$. It easy to show that for the zero capital income tax rule to hold in the short run, both (weak) separability between consumption and leisure and homotheticity in consumption are needed.

Example 1 Consider a utility function of the form

$$U(c,l) = \frac{(c-\overline{c})^{1-\sigma}}{1-\sigma} + V(l)$$

for which $\theta_{cl} = 0$ and $\dot{\theta}_c \neq 0$. This specification allows us to reinterpret the results in terms of the more traditional static problem of optimal commodity taxation, and namely of Ramsey's inverse elasticity rule. In fact, it is straightforward to show that there is an inverse relationship between the general equilibrium elasticity, H_c, and the price elasticity of consumption[16], so

[16]The steps to obtain such result are the following: first, rewrite the individual's FOC for optimal consumption by assuming that the consumption price is $q_c(t) : U_c = q_c p$.

that $\dot{H}_c = 2\frac{\left|\dot{\varepsilon}_q\right|}{[\varepsilon_q]^2}$. Consequently, when $\dot{H}_c > 0$, the elasticity of consumption is decreasing with time and equation (21) implies a positive tax on capital income[17]. In other words, this means that, since capital income taxation is a tax on future consumption, the policymaker will find it optimal to levy a positive tax whenever consumption demand is getting more inelastic. Specularly, future consumption will be subsidized when consumption demand is becoming more elastic. In any case, since consumption elasticity is constant at the steady state and since IDF has been supposed to be zero, capital taxation will be zero in the long run and the Chamley-Judd result applies.

More precisely, since with the specification above

$$\dot{\theta}_c = \frac{\dot{c}}{c}\theta_c \frac{1}{(c-\overline{c})}$$

we get, for $t > 0$:

$$\tau^k = -\frac{\frac{\dot{c}}{c}\theta_c \frac{\overline{c}}{(c-\overline{c})}\hat{\lambda}}{\left[1+\hat{\lambda}-\hat{\lambda}\theta_c\right]}.$$

Example 2 We can contrast this case with that of a utility function of the form:

Second, by applying logs and taking the derivative with respect to the price (under the assumption of separability between consumption and labor); we get:

$$\frac{U_{cc}}{U_c}\frac{\partial c}{\partial q_c} = \frac{\partial p}{\partial q_c}\frac{1}{p} + \frac{1}{q_c}.$$

Next, multiplying both members by c and rearranging terms, we obtain:

$$H_c = \left[\frac{\partial c}{\partial q_c}\frac{q_c}{c}\right]^{-1}\left[1 + \frac{\partial p}{\partial q_c}\frac{q_c}{p}\right] = \frac{1}{\varepsilon_q}\left[1+\varepsilon_{p,c}\right]$$

Now, by recalling that $p(t) = p(0)\,e^{-\int_0^t (\widetilde{r}(v)-\beta)dv}$ and that $q_c(t) = q_c(0)\,e^{-\int_0^t \widetilde{r}(v)dv}$ (and normalizing $q_c(0)$ to one), we get:

$H_c = \frac{2}{\varepsilon_q}$ and, consequently, $\dot{H}_c = 2\frac{\left|\dot{\varepsilon}_q\right|}{[\varepsilon_q]^2}$.

[17]It is worth recalling that λ is positive since it represents the social cost of resorting to distortionary taxation. Intuitively, this can be reckoned by considering that the integral constituting the implementability constraint is equal to $U_{c0}a_0$. As a consequence, reducing distortionary taxes at $t > 0$ implies increasing lump sum taxation at $t = 0$, which indeed increases welfare.

$$U(c,l) = \frac{\left(c(1-l)^{\alpha}\right)^{1-\sigma}}{1-\sigma}$$

for which $\theta_{cl} \neq 0$, $\dot{\theta}_{cl} \neq 0$ and $\dot{\theta}_c = 0$. In this case $H_{c(t)} = -\sigma + \delta \frac{l}{1-l}$, where $\delta = \alpha(1-\sigma)$, and $\dot{H}_{c(t)} = -\delta \frac{\dot{l}}{(1-l)^2}$. Thus, we get

$$\tau^k = -\frac{\delta \frac{\dot{l}}{(1-l)^2} \hat{\lambda}}{f_k \left[1 + \hat{\lambda} - \hat{\lambda}(\theta_c + \theta_{cl})\right]}.$$

Also this result can be explained with reference to the traditional optimal taxation rules, namely the Corlett and Hague (1953) one. Since leisure cannot be taxed directly, the second best solution is to tax the good that is more complementary to it. Given the specification above, consumption turns out to be complementary to leisure. Thus, as for the short run, the tax on capital income (i.e. on future consumption) will be positive when labor supply is decreasing with time (i.e. consumption and leisure are increasing).

3.2.2 The role of the time discounting rates

Let us now focus on the role played by IDF in determining the optimal tax rate on capital income. In order to do this we eliminate the effect of GEEF. Thus we can write the following:

Corollary 2 *If the utility function is homothetic in consumption and separable in consumption and labor, then the only source of capital income taxation, both in the short and in the long run, is given by IDF.*

Let us analyze separately three cases, by posing the hypothesis of separability and homotheticity of the utility function and abstracting from labor income taxation:

Case a): $\beta > \gamma$. In this case, as mentioned above, $\hat{\lambda}$ tends to zero, and, by eq. (15) $\hat{\eta}$ tends to U_c, which is constant at the steady state. This means that, according to eq. (17), $f_k = \gamma$. Now, since the steady state level of consumption must be constant as well, this means that $f_k\left(1-\tau^k\right) = \beta$. Reassembling terms, it follows that $\tau^k = 1 - \frac{\beta}{\gamma}$, which is negative. The situation can be represented by figure 1, in which $\dot{k} = 0$ and $\dot{\hat{\eta}} = 0$ are depicted and, furthermore, the locus $\dot{c} = 0$ obtaining without state intervention (i.e. $f_k = \beta$) is presented as well. As stated above, the policymaker desired level of per capita capital (k_G) is higher than the private's level (k_P). As a consequence, the policymaker will find it optimal to subsidize future consumption, in order to accelerate the rythm of capital accumulation (*i.e.* the $\dot{c}$ locus shift to the right due to the subsidy, while the $\dot{k}$ locus shifts downward due to g).

Case b. 1): $\beta < \gamma$ and $H_c \neq 1$. In this case $\hat{\lambda}$, tends to infinity, and by eq. (15), so does $\hat{\eta}$. Consequently, by observing eq. (17), candidates to be the steady state solutions of the Ramsey problem are all the levels of capital by which the condition $f_k < \gamma$ is satisfied (all the levels higher than that associated to the $f_k = \gamma$ locus). However, by observing the law of motion of $\hat{\eta}$, which reduces to: $\dot{\hat{\eta}} = \dot{\hat{\lambda}} U_c (1 + H_c)$, and dividing both sides by $\hat{\eta}$ we get: $\frac{\dot{\hat{\eta}}}{\hat{\eta}} = \frac{\dot{\hat{\lambda}} U_c (1+H_c)}{U_c\left(1+\hat{\lambda}(1+H_c)\right)}$; finally, rearranging terms, we obtain $\frac{\dot{\hat{\eta}}}{\hat{\eta}} = \frac{\dot{\hat{\lambda}}}{\hat{\lambda}\left(\frac{1}{\hat{\lambda}(1+H_c)}+1\right)}$, which is satisfied if $\gamma - f_k = \gamma - \beta$, i. e. $f_k = \beta$. In other words, this condition can be interpreted by saying that cost of the tax distortion and the opportunity cost of investment should run to infinity at the same rate. Finally, note that this condition implies, by eq. (6), that $\tau^k = 0$, i.e. the "competitive equilibrium" is achieved (point E in figure 2)[18].

Case b. 2): $\beta < \gamma$ and $H_c = -1$: it is easy to show that, when $H_c = -1$, i.e. the utility function is logarithmic, future taxation does not affect the intertemporal consumption pattern. In fact, in this case the condition $\dot{\hat{\eta}} = \dot{\hat{\lambda}} U_c (1 + H_c)$ is always satisfied if and only if $\dot{\hat{\eta}} = 0$, that is, if $\gamma = f_k$. Now, this condition implies that the stock of capital desidered by the policymaker (referred to as k_G in figure 3) is lower than the private agents' optimal level k_P (whereby $\beta = f_k$). As a consequence, at the steady state the government will find it optimal to levy a positive tax τ^k equal to $1 - \frac{\beta}{\gamma}$. Note that this happens because in the case of a utility function displaying a unitary intertemporal elasticity of substitution the substitution and the income effect generated by a

[18] In the light of the discussion above, the condition stemming from the steady state consumption growth rate, implying: $\tau^k = 1 - \frac{\beta}{\gamma}$, that is $\tau^k > 0$, would violate exactly the equality between $\hat{\eta}$ and $\hat{\lambda}$ growth rates.

future interest rate variation (due to taxation) cancel out. Hence, the change of future interest rates, that is, the change of the relative prices of future consumption, does not affect the planned allocation.

Example 3 Let us consider a utility function of the form:

$$U(c,l) = \frac{\left(c\,(l)^{-\alpha}\right)^{1-\sigma}}{1-\sigma},$$

where σ and α are positive and strictly lower than one (in order to guarantee the concavity of the utility function). It can easily be shown that both $\dot{\theta}_{cl}$ and $\dot{\theta}_c$ are equal to zero. Then, we have that the only channel of taxation, both during transition and at the steady state, is IDF. More precisely

$$\tau^k = -\frac{1}{f_k}\left[\frac{\beta-\gamma}{1+\hat{\lambda}-\hat{\lambda}\,(\sigma+\delta)}\right]$$

where $\delta = -\alpha\,(1-\sigma)$.

Now, consider first the case $\beta > \gamma$: individuals are discounting the future at a rate that is higher than the government one. As a consequence, since they are consuming at a too high rate, the government finds it optimal to subsidize capital, that is, future consumption. The same reasoning applies in the second case ($\beta < \gamma$). In both situations, however, it is worth noting that the tax rate is inversely proportional to the current value of the distortion ($\hat{\lambda}$), since increasing the capital income tax worsens the overall deadweight loss. Thus, in the first case the tax rate function will grow through time (since $\hat{\lambda}$ gets lower), while it will decrease in the other case (with $\hat{\lambda}$ getting bigger).

In the long run, when the distortion tends to zero, there is still room for subsidizing future consumption (i.e. current capital income). On the other hand, when $\hat{\lambda}$ raises exponentially to infinity, the distortionary effect overwhelms the welfare improvement due to the Pigouvian correction. Hence, the government, which cares relatively less about the present than individuals do, finds it optimal to announce a zero capital income tax for the long run.

Finally, if $\sigma = 1$, then $\delta = 0$ and the tax on capital income constant and equal to $-\frac{1}{\gamma}(\beta-\gamma)$, which is positive (negative) if the policymaker is more (less) patient than individuals.

4 Concluding remarks

By adopting the primal approach to the Ramsey problem of optimal dynamic taxation, in this paper we reconsider the conditions under which the well known Chamley-Judd zero capital tax result holds both along the transition path and at the steady state in the case of a perfectly competitive economy with infinitely lived agents and no growth. By following the insights of Judd (1999), we show that such result, while generally valid at the steady state, holds during transition only if the utility function is homothetic in consumption and separable in consumption and leisure. The technique we adopt enables us to point out the strict relationship between such result and the ones stemming from the more traditional version of the static commodity taxation problem. In fact, it turns out that, since current capital taxation is equivalent to taxing consumption in the future, it is only optimal to use such fiscal instrument if future consumption is more inelastic. In particular, if individuals' preferences are separable in consumption and leisure, we can relate the (implicit expression of the) capital income tax to the variation of the price elasticity of consumption over time (Ramsey's inverse elasticity rule). On the other hand, if the hypothesis of separability is relaxed, it turns out that the second best optimal tax rate is positive if future consumption is more complementary to leisure with respect to current one (Corlett and Hague rule).

However, we show that if the individual and government intertemporal discount rates are different, another source of taxation comes into play. In fact this factor gives room to Pigouvian correction, since, when the government weighs the present more (less) than private agents do, it is optimal to raise positive (negative) capital income taxes in order to increase (decrease) present consumption. However, in the long run the deadweight loss brought about by distortionary tax generates an asymmetry in the results: in fact, when the government is more patient, the current value of the distortion tends to zero, so that it is still optimal to subsidize future consumption; on the contrary, if the policymaker is less patient, such distortion explodes to infinity, so that the announced long run tax rate can only be zero. The asymmetry in the results is ruled out in the special case of logarithmic utility functions, which display a unitary intertemporal elasticity of substitution: in this case, since the substitution and the income effect generated by a future interest rate variation (due to taxation) cancel out and thus, the intertemporal distortionary effect of taxation does not explode, the policymaker can implement its desired policy by taxing or subsidizing accumulation, according to whether it is more or less impatient than individuals.

Concluding, it is worth noting that both our and Chamley-Judd results are obtained under the crucial hypothesis of existence of a commitment technology which can overcome the time inconsistency problem of the second best fiscal policy. Exploring the consequences of the relaxation of this assumption is a challenging avenue for future research.

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5 Appendix A: Proof of proposition 1

Proof. In this section we demonstrate that any feasible allocation satisfying implementability is a competitive equilibrium. In other words, the solution of the Ramsey problem accomplishing both feasibility and implementability constraints is a competitive equilibrium.

To start with, notice that by construction, a competitive equilibrium (or implementable allocation) satisfies also feasibility and implementability. We now show the reverse.

Suppose that an allocation satisfies implementability and the feasibility constraint. Then, define a sequence of after tax prices as follows: $\widetilde{w}(t) = -\frac{U_{l(t)}}{U_{c(t)}}$, $\widetilde{r}(t) = \left(\frac{\dot{p}(t)}{p(t)} + \beta\right)$, with $p(t) = U_{c(t)}$ $\forall t$, and a sequence of before tax prices as: $f_{k(t)} = r(t)$ and $f_{l(t)} = w(t)$. As a consequence, by construction such allocation satisfies consumer optimum and factor prices optimality.

The second step is to show that the allocation satisfies the consumer budget constraint. Take the implementability constraint and substitute $U_{c(t)}$, $U_{l(t)}$ by using the expressions above:

$$\int_0^\infty e^{-\beta t} \left[p(t) c(t) - \widetilde{w}(t) p(t) l(t)\right] dt = a(0) p(0)$$

then, by exploiting the expression for $\dot{p}(t)$ we get:

$$\int_0^{\infty} e^{-\beta t} p\left(0\right) e^{-\int_0^t (\widetilde{r}(v)-\beta)dv} \left(c\left(t\right) - \widetilde{w}\left(t\right) l\left(t\right)\right) dt = a\left(0\right) p\left(0\right)$$

$$\int_0^{\infty} e^{-\int_0^t \widetilde{r}(v)dv} \left(c\left(t\right) - \widetilde{w}\left(t\right) l\left(t\right)\right) dt = a\left(0\right)$$

Finally, by defining $c\left(t\right) - \widetilde{w}\left(t\right) l\left(t\right) = \dot{q}\left(t\right) - \widetilde{r}\left(t\right) q\left(t\right)$ we get that

$$\int_0^{\infty} \frac{d\left[q\left(t\right) e^{-\int_0^t \widetilde{r}(v)dv}\right]}{dt} = a\left(0\right)$$

which holds if $q\left(t\right) = a\left(t\right)$ and $\lim_{t\to\infty} a\left(t\right) e^{-\int_0^t \widetilde{r}(v)dv} = 0$.

Finally, as for the public sector budget constraint, by exploiting the fact that the allocation satisfies the feasibility constraint, and by recalling the homogeneity of degree one of the production function, we can write:

$$\dot{k}\left(t\right) = w\left(t\right) l\left(t\right) + r\left(t\right) k\left(t\right) - c\left(t\right) - g$$

Then, by substituting the expression for consumption obtainable by the individual budget constraint, we get:

$$\dot{k}\left(t\right) - \dot{a}\left(t\right) = w\left(t\right) l\left(t\right) + r\left(t\right) k\left(t\right) - \widetilde{w}\left(t\right) l\left(t\right) - \widetilde{r}\left(t\right) k\left(t\right) - g.$$

Finally, by defining $b\left(t\right) = k\left(t\right) - a\left(t\right)$ and exploiting the definition of taxes, the previous expression becomes:

$$\dot{b}\left(t\right) = \widetilde{r}(t)b\left(t\right) + g - \tau^k\left(t\right) r(t)k(t)\text{-}\tau^l\left(t\right) w(t)l\left(t\right)$$

■

6 Appendix B: The "isoperimetric problem"

Following Kamien and Schwartz (1981), an "isoperimetric problem" can be represented as follows:

$$\max \int_0^T F\left(t, u, y\right) dt \tag{B.1}$$

$$\text{subject to } \dot{y} = f\left(t, u, y\right)$$
$$\int_0^T R\left(t, u, y\right) dt = k,\ k \text{ given}$$
$$\text{and other boundary conditions on } y,$$

where t is time, u and y are the control and the state variable respectively. Clearly, the peculiarity of such a kind of problem is that an equality integral appears among the constraints.

However, the problem has an equivalent representation, since, by defining:

$$W\left(t\right) = \int_0^t R\left(s, u, y\right) ds$$

so that:

$$\dot{W}\left(t\right) = R\left(t, u, y\right);$$

the control problem can be written as

$$\max \int_0^T F\left(t, u, y\right) dt \tag{B.2}$$

$$\text{subject to } \dot{y} = f\left(t, u, y\right)$$
$$\dot{W} = R\left(t, u, y\right).$$
$$\text{and other boundary conditions on } y.$$
$$W\left(0\right) = 0,\ W\left(T\right) = k.$$

Thus, the Hamiltonian is:

$$J = F\left(t, u, y\right) + \eta f\left(t, u, y\right) + \lambda R\left(t, u, y\right)$$

and the following conditions from the maximum principle hold:

$$\max_u J,\ \forall\ t \in [0, T]$$

$$\dot{y} = \frac{\partial J}{\partial \eta}$$

$$\dot{\eta} = \frac{\partial J}{\partial y}$$

$$\dot{W} = \frac{\partial J}{\partial \lambda} \tag{B.3}$$

$$\dot{\lambda} = \frac{\partial J}{\partial W}$$

$$\lim_{t \to \infty} J(t) = 0.$$

Finally, it is worth noting that, since W does not enter directly the Hamiltonian, it must be that $\dot{\lambda} = \frac{\partial J}{\partial W} = 0$, so that the costate variable associated with the integral constraint is constant over time.

7 Appendix C: Capital Taxation at the starting period

In this section we briefly show that the starting period taxation of capital is different from zero. We do this for hypothesis of separability of the utility function in leisure and consumption (along with the Inada conditions).

This first step to consider is that, in case the starting value of capital tax, $\tau^k(0)$, is an instrument available to the government, it is convenient to define $W(t) = \left[\int_0^t e^{-\beta s}\left[U_{c(s)}c(s) + U_{l(s)}l(s)\right]ds - a(0)\,U_{c(0)}\right]$, so that $\dot{W}(t) = \left[e^{-\beta t}\left[U_{c(t)}c(t) + U_{l(t)}l(t)\right] - V(0)\right]$, with $V(0) = U_{c(0)}\dot{a}(0) + a(0)\,U_{c(0)c(0)}\dot{c}(0)$; thus, the current value Hamiltonian turns out to be:

$$\begin{aligned} J\left(t, \hat{\lambda}, \hat{\eta}, c(t), l(t), k(t)\right) &= U(c(t), l(t)) + \hat{\eta}(t)\left(f(k(t), l(t)) - c(t) - g\right) \\ &\quad + \hat{\lambda}(t)\left[U_{c(t)}c(t) + U_{l(t)}l(t)\right] - \lambda e^{\gamma t}V(0) \end{aligned}$$

and the FOC necessary condition associated with $c(0)$ is:

$$U_{c(0)}(1+\lambda)+\hat{\lambda}H_{c(0)}U_{c(0)}-\lambda U_{c(0)}B(0)=\hat{\eta}(0)$$

where $B(0)=\left(\frac{\partial V(0)}{\partial c(0)}\right)\frac{1}{U_{c(0)}}$.

It follows that, by differentiating the condition above with respect to time:

$$U_{c(0)c(0)}\dot{c}(0)\left[(1+\lambda)+\lambda\left(H_{c(0)}-B(0)\right)\right]+\lambda\left(\dot{H}_{c(0)}-\dot{B}(0)\right)U_{c(0)}=\dot{\hat{\eta}}(0)$$

Finally, collecting terms and exploiting condition (6), we get:

$$\tau^{k}(0)=-\frac{\lambda\left(\dot{H}_{c(0)}-\dot{B}(0)\right)}{f_{k(t)}\left[1+\lambda+\lambda\left(H_{c(0)}-B(0)\right)\right]};$$

since in general $H_{c(0)}$ is different from $B(0)$ and, moreover, $\dot{B}(0)$ will be different from zero, even when $\dot{H}_{c(0)}=0$, it follows that $\tau^{k}(0)\neq 0$.

Figures

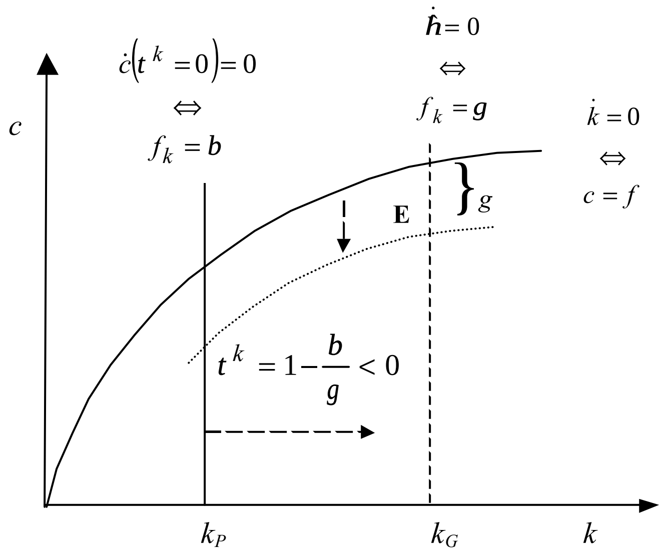

Figure 1: $b > g$

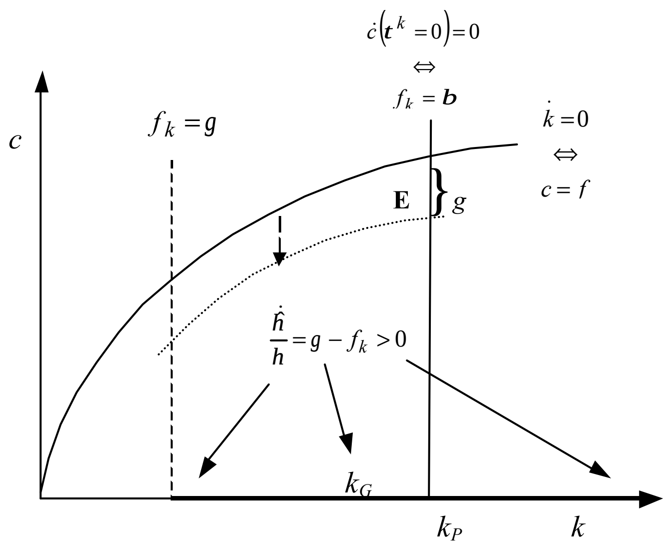

Figure 2: $b < g$ *and* $H_c \neq -1$

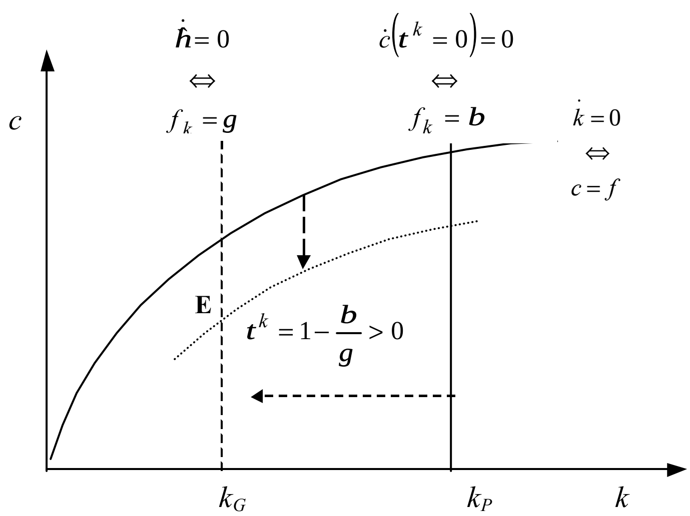

Figure 3: $b < g$ *and* $H_c = -1$