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ABSTRACT. In this paper we deal with a fair division model concerning compensation among individuals endowed with different, non transferable, personal characteristics. We construct social orderings over all allocations which rationalize the existing allocation rules and are supported by axioms that are coherent with the same ethical principles.

JEL Classification: D63, D71 D
Keywords: Compensation, fairness, social orderings

INTRODUCTION

We study the fair allocation of a given amount of some divisible resource (e.g. money) among a finite population of individuals having different, non transferable, personal characteristics. Such a personal characteristic can be either a talent (the more, the better) or a handicap (a bad) and can be exemplified by features of human capital like health condition, bodily characteristic, education, social background. The purpose of the distribution mechanism is reducing inequalities stemming from different endowments in such personal resources: individuals cannot be held responsible[1] for the amount of talent or handicap they are endowed with and this raises the question of a fair compensation.

In particular, as in Fleurbaey [4] and [5], Bossert, Fleurbaey and Van de Gaer. [3], Flaurbey and Maniquet [10] we will adhere to two general ethical principles which, using Fleurbaey's words, "express the aim of compensating *only* but *fully* for the handicap differentials of the agents".

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[1]the definition of the notion of responsibility is beyond the scope of this paper. Recent developments of egalitarian justice (Rawls 1971, Dworkin 1981, Arneson 1989, Cohen 1989, van Parijs 1995) give some possible suggestions of how responsibility could be taken into account when a fair distribution of resources is pursued

First we consider the necessity to neutralize (compensate as *fully* as possible) the differential influence over agents' outcomes of the characteristics for which they are not held responsible for. In other words, inequalities due to differences in personal characteristics should be reduced.

Second, differences due to other characteristics should be considered neutrally, or, to put it differently, if the compensation is done *only* for handicaps (talents) , then two agents with the same handicap (talent) should be treated equally.

The former principle is known as *principle of compensation* the latter as *principle of natural reward*.

The literature has proposed first best allocation rules (for a complete survey of the subject see Flaurbey and Maniquet [10]). This kind of approach, though useful, can face some practical limits. A social planner wishing to introduce some reforms can face some social or legal or political constraints which don't allow him to achieve the first best solution. Here we look at social ordering functions. More precisely, we propose axioms describing the ethical principles mentioned above and we propose some social ordering functions supporting these axioms. In this way, even if some constraints arise, it is still possible to find a feasible allocation of resources which satisfies at best the ethical principles we want to fulfill.

In accordance with Fleurbaey and Maniquet [11], [8] and [7] such social ordering functions rely only on the ordinal non comparable information on individual preferences.

We first present a negative result: we show that it is impossible to satisfy full compensation and equal reward requirements in their full extent. This impossibility leads necessarily to a choice concerning which one of the two principles we want to give priority to. In particular if we give priority to the principle of compensation then we prove that a small aversion to inequality combined with some robustness conditions leads to an infinite inequality aversion. If we consider that the resource to be shared is purely monodimensional this result is to some extent in contrast with the classical theory of inequality measurement. Finally we give the characterization of two different social ordering functions which have a symmetric behavior regarding the compensation properties (compensation only but fully)

The paper is organized as follows. In Section 1 we introduce the model and the relevant notation. In section 2 we introduce the requirements imposed on the social preferences and some preliminary results. The social ordering functions used in this paper are introduced in section 3 together with the main results. Proofs are provided in the appendix.

1. The Model

The model we analyze is derived from Fleurbaey [4], [5]. We consider a pure exchange economy, with a finite number of agents N. A given amount $\Omega \in \mathbb{R}_{++}$ has to be shared among those individuals through transfers of amount $x_i \in \mathbb{R}_+$. Each agent $i \in N$ is characterized by a parameter $y_i \in Y$, where y_i represents her personal, non transferable, resource (handicaps or talents) and Y is the set containing all possible parameters y that each agent may have. In Fleurbaey's model no particular mathematical structure is assumed for this set, we allow for ordinal comparisons between different levels of handicap. Moreover each agent is endowed with personal preferences R_i over extended bundles, i.e. over pairs $(x_i, y_i) \in \mathbb{R}_+ \times Y$. The individual preference relation R_i is assumed to be continuous and strictly monotonic with respect to x_i and y_i.

So, an economy is denoted by $\varepsilon = \{y_N; R_N; \Omega\}$, where $y_N = (y_1, ..., y_n)$ is the population profile of *talents* (i.e. the individual characteristic to be compensated), $R_N = (R_1, ..., R_n)$ is the profile of individual preferences (strict preference and indifference will be respectively denoted by P_i and I_i). Let $\mathcal{R}$ denote the set of such preferences. The domain of all the economies satisfying the above assumptions will be denoted by $\mathcal{D}$.

An allocation is a vector $x_N = (x_1, ..., x_n) \in X \subset \mathbb{R}^n_+$. It is said to be feasible if $\sum_{i \in N} x_i = \Omega$ we will indicate with $F(\varepsilon)$ the set of feasible allocations in a given economy ε. The main concern of this paper is to construct complete ordering of all the (feasible and non feasible) allocations for all economies in the domain. More formally, a social ordering function (SOF) $\overline{R}$ associates every admissible economy $\varepsilon \in \mathcal{D}$ with a complete ordering $\overline{R}(\varepsilon)$ over X. So for an economy $\varepsilon = \{y_N; R_N; \Omega\} \in \mathcal{D}$ and two allocations x_N and $x'_N \in X$ we write $x_N \overline{R}(\varepsilon) x'_N$ to denote that x_N is (socially) at least as good as x'_N. Strict social preference and indifference will be respectively denoted by $\overline{P}(\varepsilon)$ and $\overline{I}(\varepsilon)$.

2. Axioms

In this section we either introduce axioms describing properties generally considered desirable for a social ordering function and more specifical axioms related to the compensation problem and to the ethical principles listed above.

We first of all introduce the standard social choice condition of efficiency which is very intuitive in this framework and it is always satisfied by strictly feasible allocations:

Strong Pareto: *for all* $\varepsilon \in \mathcal{D}$ *and for all* $x_N, x'_N \in X$, *if for all* $i \in N$ $x_i \geq x'_i$ *then* $x_N R(\varepsilon) x'_N$; *if in addition for some* $i \in N, x_i > x'_i$ *then* $x_N \overline{P}(\varepsilon) x'_N$.

We can now turn to cross-economy robustness axioms. The first one, called Independence of Alternatives Outside Indifference Curves, is a weakening of the famous Arrow's Independence of Irrelevant Alternatives (see Fleurbaey and Maniquet [11]). It requires that the ranking of two allocations be not affected by changes in preferences that do not modify the agents' indifference curves at the bundles composing two allocations:

Independence of Alternatives Outside Indifference Curves (IAOIC): *for all* $\varepsilon \in \mathcal{D}$ *and for all* $x_N, x'_N \in X$, *if for all* $i \in N$,

$$I(R'_i, x_i) = I(R_i, x_i) \quad \textit{and} \quad I(R'_i, x'_i) = I(R_i, x'_i),$$

then $x_N \overline{R}(\varepsilon) x'_N \iff x_N \overline{R}(\varepsilon') x'_N$

The next axiom (introduced by Fleurbaey and Maniquet [7]) is logically related to the well known separability condition (see D'Aspremont and Gevers [1]) according to which, agents who are indifferent over some alternatives, should not influence social preferences over those alternatives. In this case we only consider agents whose bundles remain unchanged: removing these agents from the economy shouldn't alter the pre-existing ranking:

Separation: *for all* $\varepsilon \in \mathcal{D}$ *and for all* $x_N, x'_N \in X$, *if there is* $i \in N$ *such that* $x_i = x'_i$, *then*

$$x_N \overline{R}(\varepsilon) x'_N \iff x_{N\setminus\{i\}} \overline{R}(R_{N\setminus\{i\}}, \Omega) x'_{N\setminus\{i\}}$$

We turn now to equity properties fulfilling the ethical requirements described by the principle of compensation and the principle of natural reward. It's useful to remark, on one side, that the former one simply states that agents who differ only in personal characteristic should end-up with the same welfare level. So, consider two agents with the same preferences and a different handicap. Consider also an allocation in which one of them obtains a strictly higher welfare level. Performing a non-leaky resource transfer from the better-off agent to the other one (such that their positions in term of welfare are not reversed) must be, according to the compensation principle, a social improvement. More formally:

Equal Preferences Transfer: *for all* $\varepsilon \in \mathcal{D}$, *for all* $x_N, x'_N \in X$, *if there exist* $i, j \in N$ *such that* $R_i = R_j$ *and there exists* $\Delta \in R_+$ *s.t:* $(x'_j, y_j)P_i(x'_i, y_i)$ *where* $x'_j = x_j - \Delta$ *and* $x'_i = x_i + \Delta$, *then*

$$x'_N \overline{P}(\varepsilon) x_N$$

We also consider a quite natural strengthening of this axiom: instead of requiring that the preferences of the agents be identical, we simply require that their indifference curves trough their bundles in x_N and x'_N are nested. Let $L[(x_i, Y_i), R_i]$ and $U[(x_i, Y_i), R_i]$ denote respectively the (closed) lower contour set and the upper contour set of R_i at (x_i, y_i) than:

Nested Preferences Transfer: $\forall \varepsilon \in \mathcal{D}, \forall x_N, x'_N \in X$ *if there exist* $i, j \in N$ *such that*

$$(x'_j, y_j)P_j(x'_k, y_i)$$

$$U((x'_j, y_j), R_j) \bigcap L((z'_k, y_k), R_k) = \emptyset,$$

and there exists $\Delta \in R_+$ *s.t:* $x'_j = x_j - \Delta$ *and* $x'_i = x_i + \Delta$ *then*

$$x'_N \overline{P}(\varepsilon) x_N$$

The rationale of this axiom is that in both the allocations one of the two agents envies the other (not only given her actual handicap but also for any hypothetical level of handicap), so performing the transfer actually reduces resources inequality.

These formulations are directly inspired to the Pigou-Dalton principle and tell us how to rank two social alternatives with the final objective of reducing the degree of inequality (in terms of welfare).

On the other side the principle of natural reward says how to distribute resources among people only considering the distribution of y_i. Basically if two individuals have the same talent there is no way to justify the fact that they receive a different amount resources: in this case any transfer of equal amount leading to a reduction of such inequality should be considered a social improvement:

Equal Handicap Transfer: *for all* $\varepsilon \in \mathcal{D}$, *for all* $x_N, x'_N \in X$ *if* *there exist* $i, j \in N$ *such that* $y_i = y_j$ *and there exists* $\Delta \in R_+$ *s.t*: $x_j - \Delta = x'_j > x'_i = x_i + \Delta$, *then*

$$x'_N \overline{P}(\varepsilon) x_N$$

The last three axioms rank situations in which there is an agent that is definitively better off in both states of the world. The next two axioms are inspired by the Suppes grading principle and deal with situations in which the agents end up with symmetrical outcomes. Suppose first that there are two agents with the same preferences and two allocations such that in one case one agent is better off and in the other is worst off (and viceversa) and the welfare level they get is symmetrical. According to the compensation principle there is no reason whatsoever for which society should prefer one allocation rather than another one. More formally:

Equal Preferences Permutation: *for all* $\varepsilon \in \mathcal{D}$, *for all* $x_N, x'_N \in X$, *if there exist* $i, j \in N$ *such that* $R_i = R_j$ *and if*

$$(x_j, y_j) I_j (x'_i, y_i) \text{ and } (x_i, y_i) I_i (x'_j, y_j)$$

with $x_h = x'_h$ $\forall h \neq j \neq i$ *then*:

$$x_N \overline{I}(\varepsilon) x'_N$$

The next axiom requires, according to the *principle of natural reward* that permuting the budgets of two agents having the same handicap but possibly different preferences does not alter the value of the allocation in the social ranking:

Equal Handicap Permutation: *for all* $\varepsilon \in D'$, *for all* $x_N, x'_N \in X$, *if there exist* $i, j \in N$ *such that* $y_i = y_j$ *and*

$$x_i = x'_j \text{ and } x_j = x'_i$$

and $x_h = x'_h$ $\forall$ $h \neq i \neq j$ *then*

$$x_N \overline{I}(\varepsilon) x'_N$$

Unfortunately it's impossible to combine these axioms together in their strongest form as shown by the next two results:

Theorem 1. *No social ordering function satisfies* Equal Preferences Transfer *and* Equal Handicap Transfer.

Theorem 2. *No social ordering function satisfies* Equal Preferences Permutation *and* Equal Handicap Permutation *and* Strong Pareto.

These two results are in line with similar ones obtained by Fleurbaey [4] and Fleurbaey and Maniquet [7]. In particular, the second one shows that the incompatibility is not due to the degree of inequality aversion exhibited by the axioms but relies on the logical incompatibility between the two underlying ethical principles. In order to get some positive result some weakening of the former axioms will be necessary. This implies that we are faced with a broader ethical question. If we decide to give priority to the principle of natural reward then we will look for the weakened versions of the axioms related to the principle of compensation. In particular, we will impose above requirements only among agents having their preferences equal to some reference preference fixed arbitrarily:

$\widetilde{R}$-**Equal Preferences Transfer**: *for all* $\varepsilon \in \mathcal{D}$, *for all* $x_N, x'_N \in X$, *for at least one* $\widetilde{R}$ [2] $\in \mathcal{R}$ *if there exist* $i, j \in N$ *such that* $R_i = R_j = \widetilde{R}$ *and there exists* $\Delta \in R_+$ *s.t:* $(x'_j, y_j)\widetilde{P}(x'_i, y_i)$ *where* $x'_j = x_j - \Delta$ *and* $x'_i = x_i + \Delta$, *then*

$$x'_N \overline{P}(\varepsilon) x_N$$

$\widetilde{R}$−**Equal Preferences Permutation**: *for all* $\varepsilon \in \mathcal{D}$, *for all* $x_N, x'_N \in X$,*for at least one* $\widetilde{R} \in \mathcal{R}$, *if there exist* $i, j \in N$ *such that* $R_i = R_j = \widetilde{R}$ *and if*

$$(x_j, y_j)\widetilde{I}(x'_i, y_i) \text{ and } (x_i, y_i)\widetilde{I}(x'_j, y_j)$$

with $x_h = x'_h \ \forall h \neq j \neq i$ *then*:

$$x_N \overline{I}(\varepsilon) x'_N$$

[2]Let ϕ be a mapping, arbitrarily chosen by the planner, from $\bigcup_{n \geq 1} \mathcal{R}^n$ *to* $\mathcal{R}$; then $\widetilde{R} = \phi(R_1, ..., R_n)$

On the other side, if we want to weaken the axioms related to principle of natural reward we impose the above requirements only among agents having her handicap equal to some reference handicap fixed arbitrarily:

$\widetilde{y}$-Equal Handicap Transfer: *for all* $\varepsilon \in \mathcal{D}$, *for all* $x_N, x'_N \in X$ *for at least one* $\widetilde{y}$ [3] $\in Y$ *if there exist* $i, j \in N$ *such that* $y_i = y_j = \widetilde{y}$ *and there exists* $\Delta \in R_+$ *s.t*: $x_j - \Delta = x'_j > x'_i = x_i + \Delta$

$$x'_N P(\varepsilon) x_N$$

$\widetilde{y}-$Equal Handicap Permutation: *for all* $\varepsilon \in D'$, *for all* $x_N, x'_N \in X$ *,for at least one* $\widetilde{y} \in Y$ *if there exist* $i, j \in N$ *such that* $y_i = y_j = \widetilde{y}$ *and*

$$x_i = x'_j \text{ and } x_j = x'_i$$

and $x_h = x'_h \; \forall \; h \neq i \neq j$ *then*

$$x_N \overline{I}(\varepsilon) x'_N$$

2.1. **The fairness axioms and degree of inequality aversion.**

The equity axioms we have introduced so far can be divided two categories: on one side we have the *transfer* axioms which exhibit a mild (or anyway limited) degree of inequality aversion. On the other side we have the *permutation* axioms which are basically neutral with respect to inequality aversion (in the sense that they are consistent with any degree of inequality aversion including negative inequality aversion). In this section we will show that if we give priority the principle of compensation then, a mild egalitarian requirement, combined with anonymity conditions and some robustness condition can lead, even in a monodimensional framework like ours, to an infinite inequality aversion. In particular, transfer conditions only justify transfers with no leakage (what one gives equals what another receives); we introduce the strengthened versions either of Equal Preferences Transfer and of Nested Preferences Transfer: instead of considering the case in which the amount of resources taken from j equals the amount given to k during the transfer, we allow now for leaky-budget transfers. That is, the amount of resources taken from j is bigger (possibly much bigger) than the amount of resources added to k's bundle. If we want to give

[3]Let ψ be a mapping, arbitrarily chosen by the planner, from $\bigcup_{n\geq 1} Y^n$ *to* Y; then $\widetilde{y} = \psi(y_1, ..., y_n)$

absolute priority to the worst-off then even in in this case we should require the resulting allocation to be socially preferred to the initial one. More formally:

Equal Preferences Priority: *for all* $\varepsilon \in \mathcal{D}$*, for all* $x_N, x'_N \in X$*, if there exist* $i, j \in N$ *such that* $R_i = R_j$ *and if*

$$(x_j, y_j)P_j(x'_j, y_j)P_j(x'_i, y_i)P_i(x_i, y_i)$$

with $x_h = x'_h \; \forall h \neq j \neq i$ *then:*

$$x'_N \overline{P}(\varepsilon) x_N$$

Nested Preferences Priority: *for all* $\varepsilon \in \mathcal{D}$*, for all* $x_N, x'_N \in X$*, if there exist* $i, j \in N$ *such that*

$$(x_j, y_j)P_j(x'_j, y_j)P_j(x'_i, y_i)P_i(x_i, y_i)$$
$$U((x'_j, y_j), R_j) \bigcap L((x'_k, y_k), R_k) = \emptyset,$$

with $x_h = x'_h \; \forall h \neq j \neq i$ *then*:

$$x'_N \overline{P}(\varepsilon) x_N$$

We can now introduce the results of this section: if one is willing to accept some mild egalitarian requirements and to impose some cross-economy robustness properties then only an infinite inequality aversion is possible:

Lemma 1. *On the domain* $\mathcal{D}$*, if a SOF satisfies* IAOIC*,* Equal Preferences Permutation*, and* Equal Preferences Transfer *then it satisfies* Equal Preferences Priority*.*

Lemma 2. *On the domain* $\mathcal{D}$*, if a SOF satisfies* Separation*,* Equal Preferences Permutation*, and* Nested Preferences Transfer *then it satisfies* Nested Preferences Priority*.*

3. Social ordering Functions

In this section we try to understand how to combine together the axioms listed so far in order to construct *fair* social ordering functions. We have already shown that is impossible to combine together compensation and reward requirements in their strongest extent, but, it is still possible to define and possibly characterize some social ordering functions using some weakened version of the axioms we have seen. The first social ordering function we introduce is an adaptation of the well known Egalitarian Equivalent Social Ordering Function (Pazner and Schmeidler [16]) and has already been proposed for this framework by Bossert, Fleurbaey and Van de Gaer. [3]. So assume a reference handicap $\widetilde{y}$ is arbitrarily chosen. This social ordering function applies the leximin criterion to the individual levels of resource $\widehat{x}$ such that the agent is indifferent between her current situation, (x_i, y_i), and the one hypothetically determined by the reference talent, that is $(\widehat{x}_i, \widetilde{y})$. Such a way of ranking the alternatives gives priority to the agents with a low $\widehat{x}$, that is, either agents with a low x_i or agents that dislike their talent y_i. To put things more formally:

$\widetilde{y}$-leximin : $\forall \varepsilon \in \mathcal{D};\ \forall x_N,\ x'_N \in \mathbb{R}^N_+$

$$x_N \overline{R}(\varepsilon) x'_N \Longleftrightarrow \widehat{x}_N \geq_{lex} \widehat{x}'_N$$

where $\widetilde{y} = \psi(y_1, ..., y_n)$ and $\widehat{x}(x_i, y_i, R_i, \widetilde{y})$ is defined as a level of resources s.t.:

$$[(x_i, y_i) I_i (\widehat{x}_i, \widetilde{y})].$$

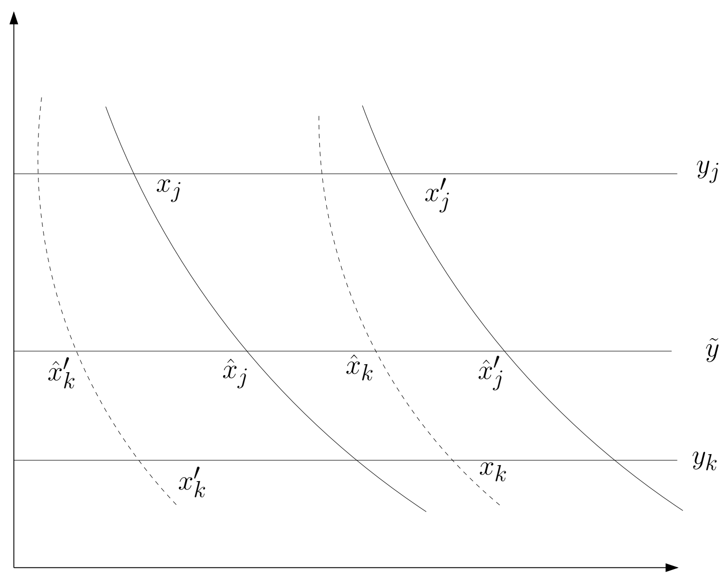

FIGURE 1

An example of how $\widetilde{y}$-**leximin** works is given in figure 1. We have a very simple economy with only two agents, j and k. We have two possible allocations $x_N = (x_j, x_k)$ and $x'_N = (x'_j, x'_k)$ and we want to rank them. The only information needed are the indifference curve of the agent at the bundles; knowing this we can easily compute the $\widetilde{y}$-equivalent valuation of such indifference curves. Finally, applying the leximin criterion to the vector of values we obtain from the evaluation process we have that $x_N \overline{P}(\varepsilon) x'_N$.

The $\widetilde{y}$-**leximin** function satisfies the strong versions of the compensation axioms. We will consider now a social ordering function which, on the contrary, fully embodies reward requirements. Basically in this case, for each allocation, the N extended bundles are ordered according to a reference preference $\widetilde{R} \in \mathcal{R}$ in such a way that $(x_i, y_i)\widetilde{R}(x_{i+1}, y_{i+1})\ \forall i \in (1, ..., n-1)$. At this point we rank the ordered vectors of bundles according to the lexicographic criterion respect to $\widetilde{R}$. More formally:

$\widetilde{R}$**-leximin** : $\forall \varepsilon \in \mathcal{D}$; $\forall x_N,\ x'_N \in \mathbb{R}^N_+$

$$x_N \overline{P}(\varepsilon) x'_N \Longleftrightarrow \textit{there exist } j \in (1, ..., n) \quad \text{s.t.} \quad (x_i, y_i)\widetilde{I}(x'_i, y_i)\ \forall i > j \quad \text{and} \quad (x_j, y_j)\widetilde{P}(x'_j, y_j)$$

on the other hand

$$x_N \overline{I}(\varepsilon) x'_N \Longleftrightarrow \forall i \in (1, ..., n) \quad (x_i, y_i)\widetilde{I}(x'_i, y_i)$$

Figure 2 gives a simple example with a two agents economy.

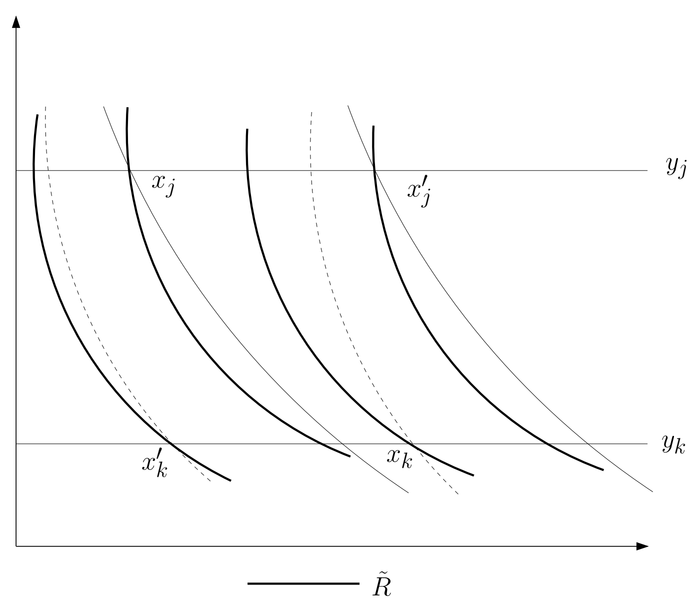

FIGURE 2

The main results of this paper may now be formulated. They are both to some extent consistent with the characterization of allocation rules in Fleurbaey [5]:

Theorem 3. *A Social Ordering Function satisfies* Strong Pareto*,* Nested Preferences Transfer*,* Equal Preferences Permutation*,*$\widetilde{y}$-Equal Handicap Permutation *and* Separation *if and only if it is a* $\widetilde{y}$-leximin *function.*

Theorem 4. *A Social Ordering Function satisfies* Strong Pareto*,* Equal Handicap Priority*,* Equal Handicap Permutation*,* $\widetilde{R}$-Equal Preferences Permutation*, and* Separation *if and only if it is a* $\widetilde{R}$*-leximin Function.*

Let us stress again the substantial asymmetry of these results: in the first one we can immediately see the consequences of lemma 3: the only axiom that embodies some inequality aversion is Nested Preferences Transfer but this is enough for the characterization of a social ordering function of the leximin type. This is not the case for the second characterization result where we can obtain a social ordering function of the leximin type only imposing a strongly egalitarian requirement

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4. Appendix

Proof of Theorem 1: *Take the economy* $\varepsilon = ((R', R', R, R), (y, y', y', y), \Omega) \in \mathcal{D}$, the points (x^a, y), (x^b, y), (x^c, y), (x^d, y), (x^e, y'), (x^f, y'), (x^g, y'), (x^h, y') and a real number Δ with:

- ***a*:** $y > y'$
- ***b*:** $(x^a, y)P(x^h, y')$
- ***c*:** $(x^e, y')P'(x^d, y)$
- ***d*:** $x^c = x^d - \Delta > x^b = x^a + \Delta$
- ***e*:** $x^g = x^h - \Delta > x^f = x^e + \Delta$

(We give a graphical example of such an economy in fig. 3)

By Equal Preferences Transfer *and conditions (b) (d) and (e) we have* (x^a, x^h, x^e, x^c) $\overline{P}(\varepsilon)$ (x^b, x^g, x^e, x^c). *By* Equal Handicap Transfer *and condition (e)* (x^a, x^g, x^f, x^c) $\overline{P}(\varepsilon)$ (x^a, x^h, x^e, x^c). *By* Equal Preferences Transfer *and conditions (c), (d) and (e)* (x^a, x^g, x^e, x^d) $\overline{P}(\varepsilon)$ (x^a, x^g, x^f, x^c) *and finally, by* Equal Handicap Transfer *and condition (d),* (x^b, x^g, x^e, x^c) $\overline{P}(\varepsilon)$ (x^a, x^g, x^e, x^d). *So by transitivity* (x^a, x^g, x^f, x^c) $\overline{P}(\widehat{\varepsilon})$ (x^a, x^g, x^f, x^c), *and we have a contradiction.* ■

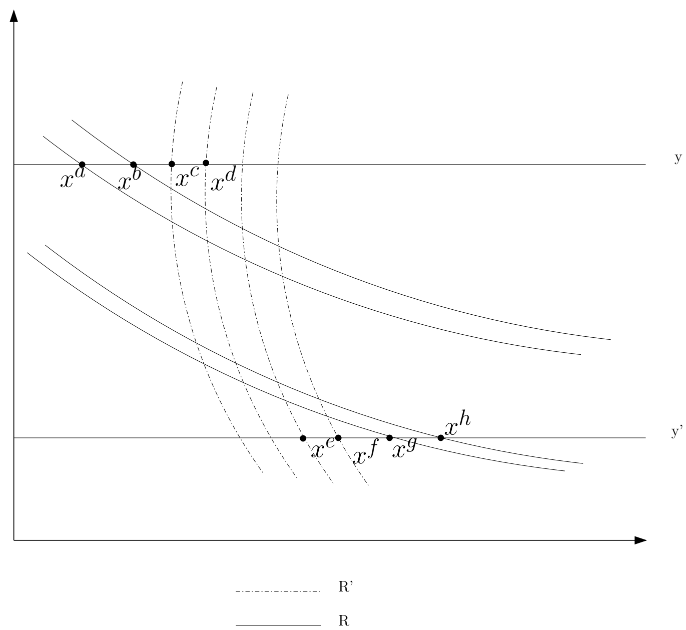

Figure 3

Proof of Theorem 2: *Take the economy* $\varepsilon = ((R, R', R', R), (y, y, y', y'), \Omega) \in \mathcal{D}$, *the points* (x^a, y), (x^b, y), (x^c, y), (x^d, y), (x^e, y), (x^f, y), $(x^{a'}, y')$, $(x^{b'}, y')$, $(x^{c'}, y')$, $(x^{d'}, y')$, $(x^{e'}, y')$, $(x^{f'}, y')$ with:

***a*:** $y > y'$
***b*:** $x^a > x^b > x^c > x^d > x^e > x^f$
***c*:** $x^{a'} > x^{b'} > x^{c'} > x^{d'} > x^{e'} > x^{f'}$
***d*:** $(x^a, y)I(x^{b'}, y')$ *and* $(x^f, y)I(x^{c'}, y')$
***e*:** $(x^b, y)I'(x^{a'}, y')$ *and* $(x^c, y)I'(x^{f'}, y')$

(such an economy is shown in figure 4).

By Strong Pareto *and conditions (b) and (c) we have* $(x^a, x^d, x^{c'}, x^{e'})$ $\overline{P}(\varepsilon)$ $(x^b, x^e, x^{d'}, x^{f'})$. *By* Equal Handicap Permutation, $(x^d, x^a, x^{c'}, x^{e'})$ $\overline{I}(\varepsilon)$ $(x^a, x^d, x^{c'}, x^{e'})$. *By* Equal Preferences Permutation *and condition (d),* $(x^d, x^f, x^{b'}, x^{e'})$ $\overline{I}(\varepsilon)$ $(x^d, x^a, x^{c'}, x^{e'})$. *By* Equal Handicap Permutation, $(x^d, x^f, x^{b'}, x^{e'})$ $\overline{I}(\varepsilon)$ $(x^d, x^f, x^{e'}, x^{b'})$. *By* Strong Pareto *and conditions (b) and (c),* $(x^c, x^e, x^{d'}, x^{a'})$ $\overline{P}(\varepsilon)$ $(x^d, x^f, x^{e'}, x^{b'})$. *By* Equal Preferences Permutation *and condition (e),* $(x^b, x^e, x^{d'}, x^{f'})$ $\overline{I}(\varepsilon)$ $(x^c, x^e, x^{d'}, x^{a'})$. *Finally by* Transitivity *we have* $(x^b, x^b, x^{d'}, x^{f'})$ $\overline{P}(\varepsilon)$ $(x^b, x^e, x^{d'}, x^{f'})$, *a contradiction* ■.

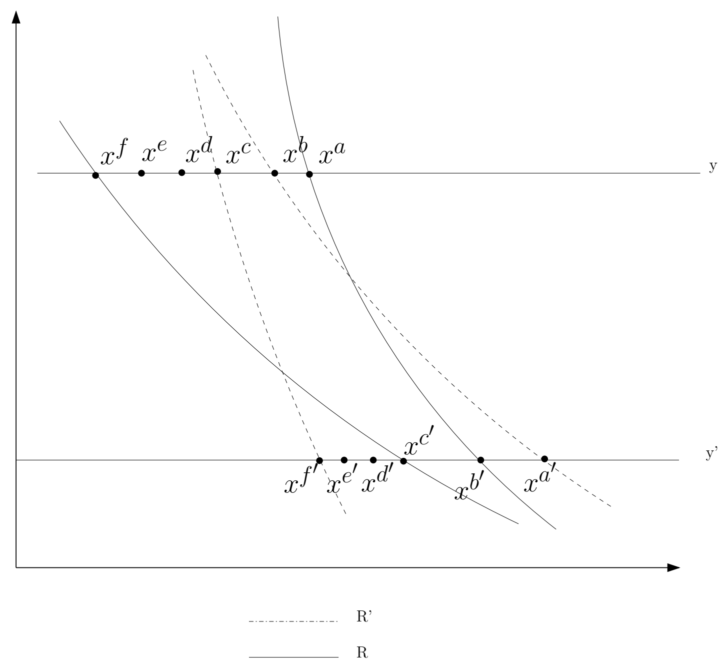

FIGURE 4

Proof of Lemma 1*: Let* $\overline{R}$ *satisfy* Separation, Equal Preferences Permutation *and* Nested Preferences Transfer. *Let* $\varepsilon = (R_N, \Omega) \in \mathcal{D}$, $x_N, x'_N \in X^n$, $j, k \in N$ *be such that:*

$$(x_j, y_j)P_j(x'_j, y_j)P_j(x'_k, y_i)P_i(x_k, y_i),$$

$$U(z'_j, R_j) \bigcap L(z'_k, R_k) = \emptyset,$$

and for all $i \neq j, k$, $x_i = x'_i$. *Let* $\triangle_j = x_j - x'_j$ *and* $\triangle_k = (x'_k - x_k)$.

Let $l, m \in \mathbb{N}_{++} \setminus N$, $R_l, R_m \in \mathcal{R}$,$y_l, y_m)$, $x_l^1, x_l^2, x_m^1, x_m^2 \in X$ and $q \in \mathbb{N}_{++}$, *be defined in such a way that* $(x_l^1, y_l) I_l (x_m^1, y_m)$ *and* $(x_l^2, y_l) I_m (x_m^2, y_m)$; $R_l = R_m$; $x_l^2 = x_l^1 + \frac{\triangle_k}{q}$ *and* $x_m^2 = x_m^1 + \frac{\triangle_j}{q}$; $(x_l^2, y_l) P_l (x_k^1, y_k)$, $U((x_l^2, y_l), R_j) \bigcap L((x'_k, y_k), R_k) = \emptyset$ $(x'_j, y_j) P_j (x_l^1, y_l)$ *and* $U((x'_j, y_j), R_j) \bigcap L((x'_l, y_l), R_l) = \emptyset$. *Let* $\varepsilon' = (R_N, R_l, R_m, \Omega) \in \mathcal{D}$ *(an example of such a construction is given in figure 5).*

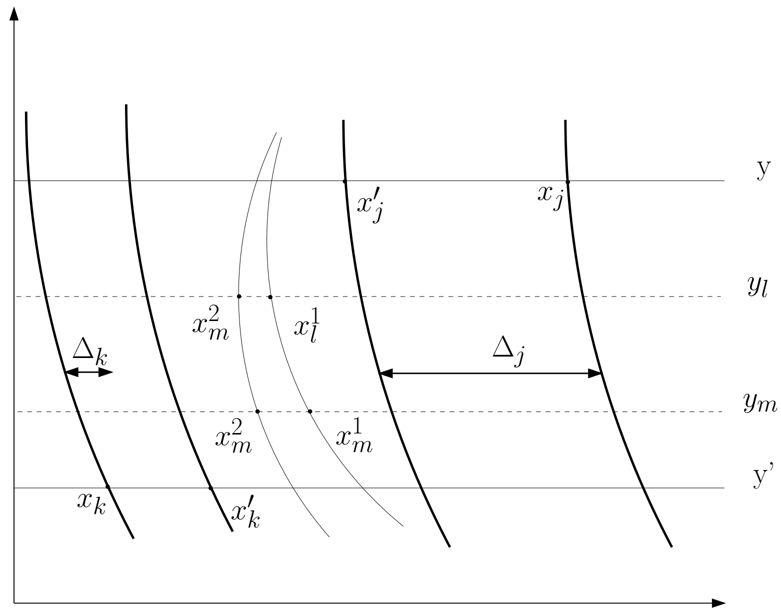

FIGURE 5

By Nested Preferences Transfer

$$(x_{N\setminus\{k\}}, x_k + \frac{\triangle_k}{q}, x_l^2, x_m^1)\overline{P}(\varepsilon')(x_N, x_l^1, x_m^1)$$

By Equal Preferences Permutation*:*

$$(x_{N\setminus\{k\}}, x_k + \frac{\triangle_k}{q}, x_l^1, x_m^2)\overline{I}(\varepsilon')(x_{N\setminus\{k\}}, x_k + \frac{\triangle_k}{q}, x_l^2, x_m^1)$$

By Nested Preferences Transfer:

$$(x_{N\setminus\{k,j\}}, x_k + \frac{\triangle_k}{q}, x_j - \frac{\triangle_j}{q}, x_l^1, x_m^1)\overline{P}(\varepsilon')(x_{N\setminus\{k\}}, x_k + \frac{\triangle_k}{q}, x_l^1, x_m^2)$$

By Transitivity:

$$(x_{N\setminus\{k,j\}}, x_k + \frac{\triangle_k}{q}, x_j - \frac{\triangle_j}{q}, x_l^1, x_m^1)\overline{P}(\varepsilon')(x_N, x_l^1, x_m^1)$$

Replicating the argument q times we get:

$$(x_{N\setminus\{k,j\}}, x_k + \triangle_k, x_j - \triangle_j, x_l^1, x_m^1)\overline{P}(\varepsilon')(x_N, x_l^1, x_m^1)$$

but $x_j - \triangle_j = x'_j$ *and* $x_k + \triangle_k = x'_k$ *so, replacing into the former equation, we get:*

$$(x'_N, x_l^1, x_m^1)\overline{P}(\varepsilon')(x_N, x_l^1, x_m^1)$$

finally, by Separation*:*

$$(x'_N)\overline{P}(\varepsilon')(x_N)$$

which is the desired result ■.

Proof of Lemma 2: *Let* $\overline{R}$ *satisfy IAOIC,* Equal Preferences Permutation *and* Equal Preferences Transfer*. Let* $\varepsilon = (R_N, \Omega) \in \mathcal{D}$*,* $x_N, x'_N \in X$ *and* $j,k \in N$ *be such that* $R_j = R_k$ *and*

$$(x_j, y_j)P_j(x'_j, y_j)P_j(x'_k, y_i)P_i(x_k, y_i),$$

with $x_i = x'_i$ *for all* $i \neq j,k$*. Let* $R'_j = R'_k \in \mathcal{R}, x_j^1, x_j^2, x_j^3, x_k^1, x_k^2, x_k^3 \in X$*,* $\triangle \in \mathbb{R}_{++}$ *be constructed in such a way that for* $i \in \{j,k\}$*,* $I((x_i, y_i), R'_i) = I((x_i, y_i), R_i)$*,* $I((x'_i, y_i), R'_i) = I((x'_i, y_i), R_i)$*,* $(x_k^1, y_k)I_i(x_j^2, y_j)$*,* $(x_k^2, y_k)I_i(x_j^1, y_j)$*,* $(x'_k, y_k)I_i(x_j^3, y_j)$*,* $(x_k^3, y_k)I_i(x'_j, y_j)$*,* $(x_j^1, y_j)P_i(x_k^1, y_k)$ *with* $x_j^1 = x_j - \Delta$ *and* $x_k^1 = x_k + \Delta$*,* $(x_k^3, y_k)P_i(x_j^3, y_j)$ *with* $x_j^2 = x_j^3 - \Delta$ and $x_k^2 = x_k^3 + \Delta$ *(see figure 6 for an example). Let* $\varepsilon' = ((R_{N\setminus\{i,j\}}, R'_j, R'_k), \Omega) \in \mathcal{D}$*.*
By Equal Preferences Transfer*:*

$$(x_{N\setminus\{j,k\}}, x_k^1, x_j^1)\overline{P}(\varepsilon')x_N$$

By Equal Preferences Permutation*:*

$$(x_{N\setminus\{j,k\}}, x_k^1, x_j^1)I(\varepsilon')(x_{N\setminus\{j,k\}}, x_k^2, x_j^2)$$

By Equal Preferences Transfer*:*

$$(x_{N\setminus\{j,k\}}, x_k^3, x_j^3)\overline{P}(\varepsilon')(x_{N\setminus\{j,k\}}, x_k^2, x_j^2)$$

By Equal Preferences Permutation*:*

$$(x_{N\setminus\{j,k\}}, x_k^3, x_j^3)\overline{I}(\varepsilon')(x_{N\setminus\{j,k\}}, x'_k, x'_j)$$

So by Transitivity $x'_N\overline{P}(\varepsilon')x_N$ *and finally by* IAOIC $x'_N P(\varepsilon)x_N$ ■

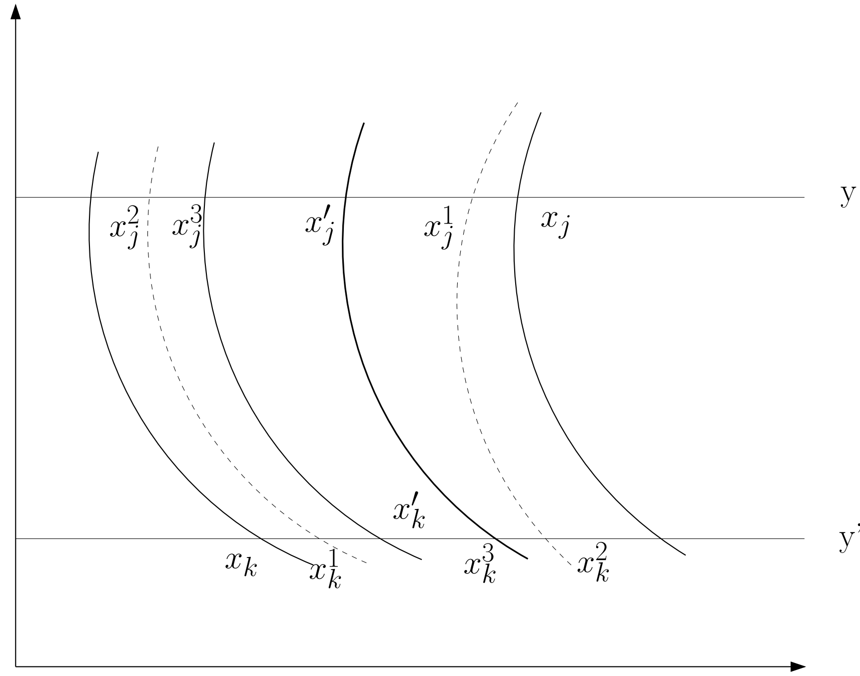

FIGURE 6

Proof of Theorem 3 *(we only prove the characterization part): Let $\overline{R}$ be a social ordering function that satisfies the listed axioms, and choose any $\widetilde{y} \in \mathbb{R}_+$. By Lemma 1 the social ordering function $\overline{R}$ also satisfies* Nested Preferences Priority.

Step 1*. Consider two allocations x_N and x'_N and two different agents j and k such that for all $i \neq j, k$, $x_i = x'_i$ Let*

$$\widehat{x}_m = min\{\widehat{x}(x_j, y_j, R_j, \widetilde{y}), \widehat{x}(x'_j, y_j, R_j, \widetilde{y}), \widehat{x}(x'_k, y_k, R_k, \widetilde{y})\}$$

Without loss of generality let us assume that $\widehat{x}(x_k, y_k, R_k, \widetilde{y}) < \widehat{x}_m$, we want to prove that $x'_N \overline{P}(\varepsilon) x_N$. If we also have $x'_j \geqslant x_j$ then by Strong Pareto *we immediately get the desired result. Consider now the case $x_j > x'_j$ and assume, by contradiction, that $x_N \overline{R}(\varepsilon) x'_N$.*

Consider the economy $\varepsilon' = (y_j, y_k, y_a, y_b; R_j, R_k, R_a, R_b; \Omega)$ and x_a, x_b, $x'_a, x'_b, x''_k \in X$ such that (see figure 7):

$$y_a = y_b = \widetilde{y} \tag{4.1}$$

$$x_a = x'_b \text{ and } x'_a = x_b \tag{4.2}$$

$$\widehat{x}(x_k, y_k, R_k, \widetilde{y}) < x_a < x_b < \widehat{x}_m \tag{4.3}$$

$$(x'_b, y_b) P_b (x''_k, y_k) \text{ and } x''_k > x_k \tag{4.4}$$

$$U((x'_j, y_j), R_j) \bigcap L((x'_a, y_a), R_a) \tag{4.5}$$

$$U((x'_b, y_b), R_b) \bigcap L((x''_k, y_k), R_k) \tag{4.6}$$

Since we have assumed $x_N \overline{R}(\varepsilon) x'_N$ *then, by* Separation, *we have* $(x_j, x_k)\ \overline{R}(y_j, y_k; R_j, R_k; \Omega)\ (x'_j, x'_k)$. *By* Separation *again* (x_j, x_k, x_a, x_b) $\overline{R}(\varepsilon')$ (x'_j, x'_k, x_a, x_b). *By* Nested Preferences Priority *and conditions* 2, 3 *and* 5 (x'_j, x_k, x'_a, x_b) $\overline{P}(\varepsilon')$ (x_j, x_k, x_a, x_b). *Again by* Nested Preferences Priority *and conditions* 2, 3, 4 *and* 6 $(x'_j, x''_k, x'_a, x'_b)$ $\overline{P}(\varepsilon')$ (x'_j, x_k, x'_a, x_b). *By* Strong Pareto *and condition* 3 (x'_j, x'_k, x'_a, x'_b) $\overline{P}(\varepsilon')$ $(x'_j, x''_k, x'_a, x'_b)$. *By* Transitivity (x'_j, x'_k, x'_a, x'_b) $\overline{P}(\varepsilon')$ (x'_j, x'_k, x_a, x_b), *which, given condition* 2 *is a violation of* $\widetilde{y}$-Equal handicap Permutation *and yields the desired contradiction so that* $x'_N \overline{P}(\varepsilon) x_N$

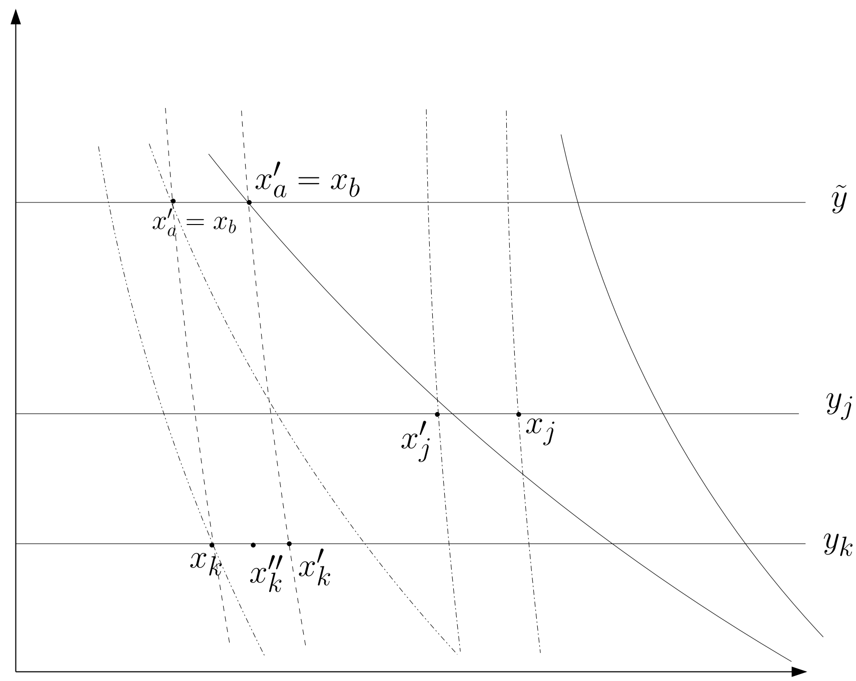

Figure 7

Step 2. *Consider two allocations* x_N *and* x'_N *and two different agents* j *and* k *such that for all* $i \neq j, k$, $x_i = x'_i$ *Let*

$$\widehat{x}(x_j, y_j, R_j, \widetilde{y}) = \widehat{x}(x'_k, y_k, R_k, \widetilde{y}) < \widehat{x}(x'_j, y_j, R_j, \widetilde{y}) = \widehat{x}(x_k, y_k, R_k, \widetilde{y})$$

Assume $x_N \overline{P} x'_N$. *Consider the economy* $\varepsilon' = (y_j, y_k, y_a, y_b; R_j, R_k, R_a, R_b; \Omega)$ *and* $x_a, x_b,\ x'_a, x'_b, \in X$ *such that (see figure 8 for an example):*

$$y_a = y_b = \widetilde{y} \tag{4.7}$$

$$R_a = R_j \text{ and } R_b = R_k \tag{4.8}$$

$$(x_a, y_a) I_j (x_j, y_j), (x'_a, y_a) I_j (x'_j, y_j), \tag{4.9}$$

$$(x_b, y_b) I_k (x_k, y_k), (x'_b, y_b) I_k (x'_k, y_k).$$

Applying Separation *twice we obtain respectively* $(x_j, x_k)\ \overline{P}(y_j, y_k; R_j, R_k; \Omega)\ (x'_j, x'_k)$ *and* $(x_j, x_k, x'_a, x'_b)\ \overline{P}(\varepsilon')\ (x'_j, x'_k, x'_a, x'_b)$. *By* Equal Preferences Permutation *and condition* 8 *and* 9 $(x'_j, x_k, x_a, x'_b)\ \overline{I}(\varepsilon')\ (x_j, x_k, x'_a, x'_b)$ *and* $(x'_j, x'_k, x_a, x_b)\ \overline{I}(\varepsilon')\ (x'_j, x_k, x_a, x'_b)$. *Finally by* Transitivity $(x'_j, x'_k, x_a, x_b)\ \overline{P}(\varepsilon')\ (x'_j, x'_k, x'_a, x'_b)$ *which violates* $\widetilde{y}$-Equal Handicap Permutation *given conditions* 7 *and* 9. *So* $x'_N\ \overline{R}(\varepsilon')\ x_N$ *and since the argument can be applied symmetrically with respect to* j *and* k *then* $x_N\ \overline{I}(\varepsilon)\ x'_N$.

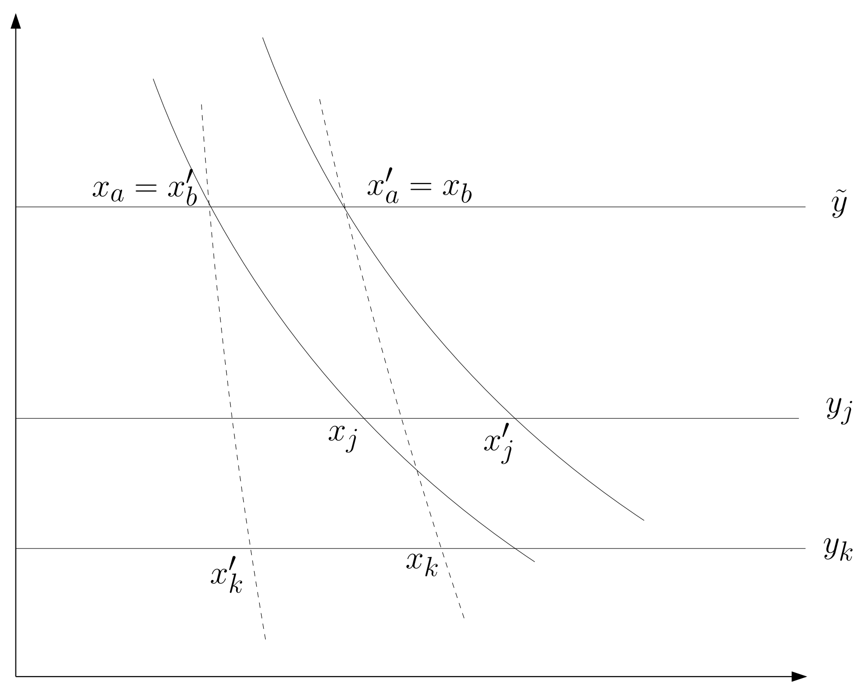

FIGURE 8

Step 3. *Consider two different allocations* x_N *and* x'_N *such that* $(\widehat{x}(x_i, y_i, R_i, \widetilde{y}))_{i\in N} =_{lex} (\widehat{x}(x'_i, y_i, R_i, \widetilde{y}))_{i\in N}$. *There exists then a permutation* π *on* N *such that for all* $i \in N$ $\widehat{x}(x_{\pi(i)}, y_{\pi(i)}, R_{\pi(i)}, \widetilde{y}) = \widehat{x}(x'_i, y_i, R_i, \widetilde{y})$. *But any permutation can be obtained by a finite number of transposition so, applying recursively step two we obtain* $x'_N \overline{I}(\varepsilon) x_N$.

Step 4. *Take now two allocations* x_N *and* x'_N *such that* $(\widehat{x}(x_i, y_i, R_i, \widetilde{y}))_{i\in N} >_{lex} (\widehat{x}(x'_i, y_i, R_i, \widetilde{y}))_{i\in N}$. *Choose two allocations* $\overline{x}$ *and* $\overline{x}' \in X$ *such that:*

$$(\widehat{x}(\overline{x}_i, y_i, R_i, \widetilde{y}))_{i\in N} =_{lex} (\widehat{x}(x_i, y_i, R_i, \widetilde{y}))_{i\in N}$$

$$(\widehat{x}(\overline{x}'_i, y_i, R_i, \widetilde{y}))_{i\in N} =_{lex} (\widehat{x}(x'_i, y_i, R_i, \widetilde{y}))_{i\in N}$$

and, such that for all $i, j \in N$ *with* $i < j$

$$\widehat{x}(\overline{x}_i, y_i, R_i, \widetilde{y}) \leq \widehat{x}(\overline{x}_j, y_j, R_j, \widetilde{y})$$

$$\widehat{x}(\overline{x}'_i, y_i, R_i, \widetilde{y}) \leq \widehat{x}(\overline{x}'_j, y_j, R_j, \widetilde{y})$$

By Step 3 we have that $x_N \overline{I}(\varepsilon) \overline{x}_N$ and $x'_N \overline{I}(\varepsilon) \overline{x}'_N$.

Furthermore, by assumption, $(\widehat{x}(\overline{x}_i, y_i, R_i, \widetilde{y}))_{i\in N} >_{lex} (\widehat{x}(\overline{x}'_i, y_i, R_i, \widetilde{y}))_{i\in N}$ *so, by construction, there must be a* $k \in N$ *such that:*

$$\begin{cases} \overline{x}_k > \overline{x}'_k & \\ (\widehat{x}(\overline{x}_i, y_i, R_i, \widetilde{y}) = (\widehat{x}(\overline{x}'_i, y_i, R_i, \widetilde{y}) & for\ all i < k \\ (\widehat{x}(\overline{x}_i, y_i, R_i, \widetilde{y}) < (\widehat{x}(\overline{x}'_j, y_j, R_j, \widetilde{y}) & for\ all j > k \end{cases}$$

Consider now the set $M = \{j \in N | \ \overline{x}'_j > \overline{x}_j\}$. *If this set is empty, then, by* Strong Pareto *we immediately have* $\overline{x}_N P(\varepsilon) \overline{x}'_N$ *and so* $x_N P(\varepsilon) x'_N$. *If* M *is not empty then assign a number from* 1 *to* $|M|$ *to each agent in* M, *each one denoted* $m(j)$. *Build then* $|M|$ *intermediate bundles* $x_N^{(n)}$, *with* $n = 1, ..., |M|$, *such that:*

$$\begin{cases} x_j^{(n)} = \overline{x}'_j & for\ all\ j \in N \setminus (M \bigcup \{k\}) \\ x_j^{(n)} = \overline{x}'_j & for\ all\ j \in M\ s.t.\ m(j) > n \\ x_j^{(n)} = \overline{x}_j & for\ all\ j \in M\ s.t.\ m(j) \leq n \end{cases}$$

and such that, the k_{th} *element in each of the intermediate bundles is chosen in the following way:*

$$min\{\widehat{x}(\overline{x}_i, y_i, R_i, \widetilde{y}) | i \in M \bigcup \{k\}\} > \widehat{x}(x_i^{(|L|)}, y_i, R_i, \widetilde{y}) > \dots$$

$$\cdots \widehat{x}(x_i^{(1)}, y_i, R_i, \widetilde{y}) > \widehat{x}(\overline{x}'_k, y_k, R_k, \widetilde{y}).$$

Applying repeatedly Step 1 we have that:

$$x_N^{|M|}\overline{P}(\varepsilon)x_N^{|M|-1}\dots x_N^1\overline{P}(\varepsilon)\overline{x}'_N$$

By Strong Pareto $\overline{x}_N\ \overline{P}(\varepsilon)\ x_N^{|M|}$ *and by* Transitivity $\overline{x}_N\overline{P}(\varepsilon)\overline{x}'_N$, *so* $x_N\overline{P}(\varepsilon)x'_N$

We finally show that no axiom is redundant:

1. *Drop Strong Pareto: replace the leximin criterion with the lexicographic minimax*[4] *over the vectors* $\widehat{x}_N$
2. *Drop Nested Preferences Transfer:* $\widetilde{y}$*-utilitarian social ordering function: the social value of any allocation* x_N *is given by* $\sum_{i=1}^N \widehat{x}_i$
3. *Drop Equal Preferences Permutation: consider a social ordering function* $\overline{R}$ *such that: for all* $\varepsilon \in \mathcal{D}$, *for all* $x_N, x'_N \in X$ *if*

$$\begin{cases}(\widehat{x}(x_i,y_i,R_i,\widetilde{y}))_{i\in N} >_{lex} (\widehat{x}(x'_i,y_i,R_i,\widetilde{y}))_{i\in N} \quad or\ if\\ (\widehat{x}(x_i,y_i,R_i,\widetilde{y}))_{i\in N} =_{lex} (\widehat{x}(x'_i,y_i,R_i,\widetilde{y}))_{i\in N}\end{cases}$$

and there exist two agents k and k' with $y_k < y_{k'}$ *such that*

$$\begin{cases}for\ all\ i\neq k\ \widehat{x}(x_k,y_k,R_k,\widetilde{y}) < \widehat{x}(x_i,y_i,R_i,\widetilde{y})\\ for\ all\ i\neq k'\ \widehat{x}(x'_{k'},y_{k'},R_{k'},\widetilde{y}) < \widehat{x}(x'_i,y_i,R_i,\widetilde{y})\end{cases}$$

then

$$x_n\overline{P}(\varepsilon)x'_n$$

In all other cases

$$x_n\overline{I}(\varepsilon)x'_n$$

4. *Drop* $\widetilde{y}$*-Equal Handicap Permutation: consider a social ordering function* $\overline{R}$ *such that: for all* $\varepsilon \in \mathcal{D}$, *for all* $x_N, x'_N \in X$ *if*

$$\begin{cases}(\widehat{x}(x_i,y_i,R_i,\widetilde{y}))_{i\in N} >_{lex} (\widehat{x}(x'_i,y_i,R_i,\widetilde{y}))_{i\in N} \quad or\ if\\ (\widehat{x}(x_i,y_i,R_i,\widetilde{y}))_{i\in N} =_{lex} (\widehat{x}(x'_i,y_i,R_i,\widetilde{y}))_{i\in N}\end{cases}$$

and there exist two agents k and k' with $R_k \succ R_{k'}$, *with* $\succ$ *being an asymmetric ordering, such that*

$$\begin{cases}for\ all\ i\neq k\ \widehat{x}(x_k,y_k,R_k,\widetilde{y}) < \widehat{x}(x_i,y_i,R_i,\widetilde{y})\\ for\ all\ i\neq k'\ \widehat{x}(x'_{k'},y_{k'},R_{k'},\widetilde{y}) < \widehat{x}(x'_i,y_i,R_i,\widetilde{y})\end{cases}$$

[4] A distribution is as good as another one if its maximum is lower, if they are equal the second highest values are compared and so on

then

$$x_n \overline{P}(\varepsilon) x'_n$$

In all other cases

$$x_n \overline{I}(\varepsilon) x'_n$$

(5) *Drop Separation: consider a social ordering function* $\overline{R}$ *such that:*

$$\overline{R}(\varepsilon) = \begin{cases} \widetilde{y} - leximin & for\ \varepsilon = \widehat{\varepsilon} \in \mathcal{D} \mid there\ exist\ i \in N\ with\ y_i = \widetilde{y} \\ \widetilde{y}' - leximin & otherwise\ (with\ \widetilde{y} \neq \widetilde{y}') \end{cases}$$

■.

Proof of Theorem 4 *(we only prove the characterization part): Let* $\overline{R}$ *be a social ordering function that satisfies the listed axioms, and choose any* $\widetilde{R} \in \mathcal{R}$ *such that it satisfies* $\widetilde{R}$-Equal Preferences Permutation. *Also in this case the proof is divided in several steps.*

Step 1*. Consider two allocations* x_N *and* x'_N *and two different agents* j *and* k *such that for all* $i \neq j, k$, $x_i = x'_i$. *Let*

$$\widetilde{X} = \{(x_j, y_j); (x'_j, y_j); (x'_k, y_k)\}$$

be the set of extended bundles relative x_j, x'_j *and* x_k. *Among these define* $(\overline{x}, \overline{y})$ *as the least preferred extended bundle with respect* $\widetilde{R}$. *Without loss of generality assume*

$$(\overline{x}, \overline{y}) \overline{P}(\varepsilon)(x_k, y_k)$$

If $x'_j \geq x_j$ *then by* Strong Pareto $x'_N \overline{P}(\varepsilon) x_N$. *Consider now the case* $x_j > x'_j$ *and assume* $x_N \overline{R}(\varepsilon) x'_N$. *Take the the economy* $\varepsilon' = (y_j, y_k, y_a, y_b; R_j, R_k, R_a, R_b; \Omega)$ *and* $x_a, x_b, x'_a, x'_b, \in X$ *such that (see figure 9 for an example):*

$$y_a = y_j \text{ and } y_b = y_k \tag{4.10}$$

$$R_a = R_b = \widetilde{R} \tag{4.11}$$

$$x'_k > x_b > x'_b > x_k \tag{4.12}$$

$$(x_a, y_a)\widetilde{I}(x'_b, y_b) \text{ and } (x'_a, y_a)\widetilde{I}(x_b, y_b) \tag{4.13}$$

By Separation, *we have* $(x_j, x_k)\ \overline{R}(y_j, y_k; R_j, R_k; \Omega)\ (x'_j, x'_k)$. *By* Separation *again* $(x_j, x_k, x_a, x_b)\ \overline{R}(\varepsilon')\ (x'_j, x'_k, x_a, x_b)$. *By* Strong Pareto $(x_j, x'_b, x_a, x'_k)\ \overline{P}(\varepsilon')\ (x_j, x_k, x_a, x_b)$, *by* Equal Handicap Priority $(x'_j, x'_b, x'_a, x'_k)\ \overline{P}(\varepsilon')\ (x_j, x'_b, x_a, x'_k)$, *by* Equal Handicap Permutation $(x'_j, x'_k, x'_a, x'_b)\ \overline{I}(\varepsilon')\ (x'_j, x'_b, x'_a, x'_k)$ *so by transitivity* (x'_j, x'_k, x'_a, x'_b)

$\overline{P}(\varepsilon')$ (x'_j, x'_k, x_a, x_b) *which is a violation of* $\widetilde{R}$-Equal Preferences Permutation*. So also in this case we have* $x'_N \overline{P}(\varepsilon) x_N$*.* ***Step 2****. Consider*

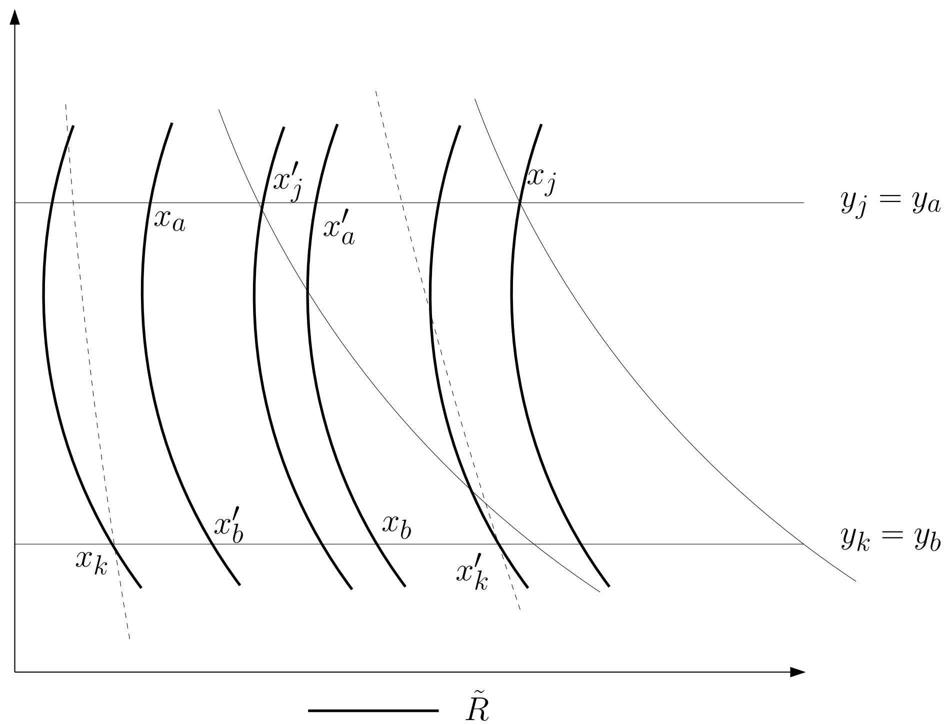

FIGURE 9

two allocations x_N *and* x'_N *and two different agents* j *and* k *such that for all* $i \neq j, k$, $x_i = x'_i$*. Let*

$$(x_j, y_j)\widetilde{I}(x'_k, y_k) \text{ and } (x'_j, y_j)\widetilde{I}(x_k, y_k)$$

with $x_j \neq x'_j$*. Consider the economy* $\varepsilon' = (y_j, y_k, y_a, y_b; R_j, R_k, R_a, R_b; \Omega)$ *and* $x_a, x_b, x'_a, x'_b, \in X$ *such that (see figure 10 for an example):*

$$y_a = y_j \text{ and } y_b = y_k \tag{4.14}$$

$$R_a = R_b = \widetilde{R} \tag{4.15}$$

$$x_a = x'_j;\ x'_a = x_j;\ x_b = x'_k;\ x'_b = x_k \tag{4.16}$$

Assume moreover $x'_N \overline{P}(\varepsilon) x_N$*. So by* Separation *we have* (x'_j, x'_k) $\overline{P}(y_j, y_k; R_j, R_k; \Omega)$ (x_j, x_k) *and* (x'_j, x'_k, x'_a, x'_b) $\overline{P}(\varepsilon')$ (x_j, x_k, x'_a, x'_b)*. By* Equal Handicap Permutation *and conditions 14, 15 and 16* (x'_a, x'_b, x'_j, x'_k) $\overline{I}(\varepsilon')$ (x'_j, x'_k, x'_a, x'_b)*. By Condition 16* (x'_a, x'_b, x'_j, x'_k) $\overline{I}(\varepsilon')$ (x_j, x_k, x_a, x_b)

and by Transitivity $(x_j, x_k, x_a, x_b)\ \overline{P}(\varepsilon')\ (x_j, x_k, x'_a, x'_b)$ *which constitutes a violation of* $\widetilde{R}$-Equal Preferences Permutation *in view of conditions 14, 15 and 16. So* $x_N\ \overline{R}(\varepsilon')\ x'_N$ *and since the argument can be applied symmetrically on the other way round then* $x_N\ \overline{I}(\varepsilon)\ x'_N$.

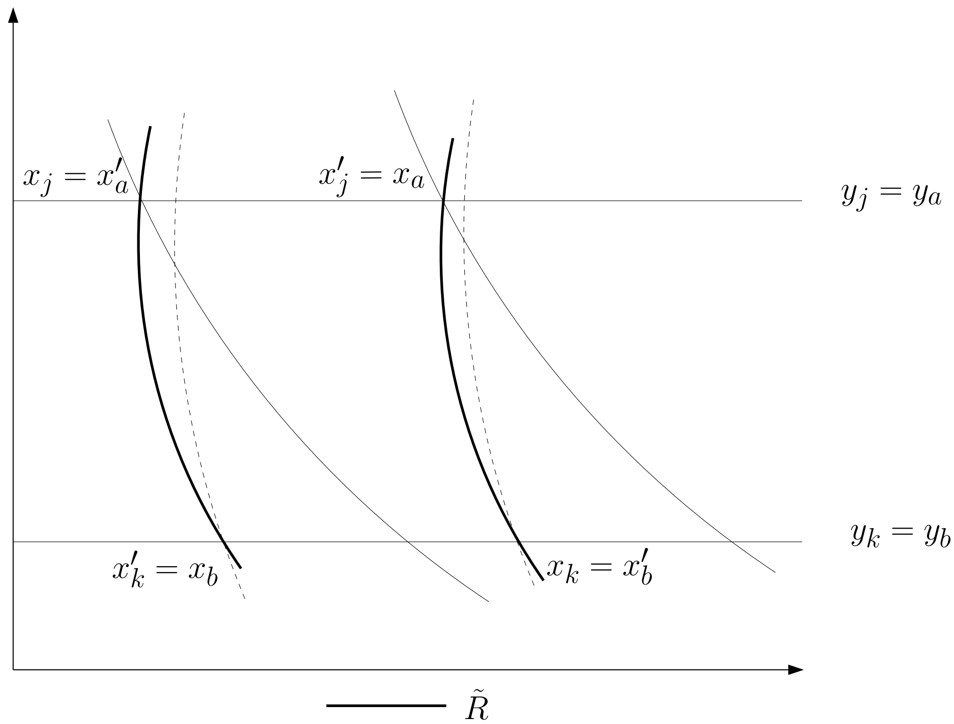

FIGURE 10

The rest of the proof parallels steps 3 and 4 of the former proof. We have just to show that no axiom is redundant:

(1) *Drop Strong Pareto: also in this case we can replace the leximin with the minimax criterion over the allocations evaluated with the reference preference* $\widetilde{R}$.
(2) *Drop Equal Handicap priority:* $\widetilde{R}$*-utilitarian social ordering function; consider some continuous utility function* $\widetilde{u}$ *representing* $\widetilde{R}$*. For any* $\varepsilon \in \mathcal{D}$ *and for any* $x_N, x'_N \in X$

$$x_N\overline{R}(\varepsilon)x'_N \Longleftrightarrow \sum_{i=1}^{N} \widetilde{u}(x_i) \geq \sum_{i=1}^{N} \widetilde{u}(x'_i)$$

(3) *Drop* $\widetilde{R}$*-Equal Preferences Permutation: consider a social ordering function* $\overline{R}$ *such that: for all* $\varepsilon \in \mathcal{D}$*, for all* $x_N, x'_N \in X$

if

$$\begin{cases} x_N \widetilde{P}_{lex} x'_N & or\ if \\ x_N \widetilde{I}_{lex} x'_N \end{cases}$$

and there exist two agents k and k' with $y_k < y_{k'}$ *such that*

$$\begin{cases} for\ all\ i \neq k & x_i > x_k \\ for\ all\ i \neq k' & x'_i > x'_{k'} \end{cases}$$

then

$$x_n \overline{P}(\varepsilon) x'_n$$

In all other cases

$$x_n \overline{I}(\varepsilon) x'_n$$

(4) *Drop Equal Handicap Permutation: consider a social ordering function* $\overline{R}$ *such that: for all* $\varepsilon \in \mathcal{D}$, *for all* $x_N, x'_N \in X$ *if*

$$\begin{cases} x_N \widetilde{P}_{lex} x'_N & or\ if \\ x_N \widetilde{I}_{lex} x'_N \end{cases}$$

and there exist two agents k and k' with $R_k \succ R_{k'}$, *with* $\succ$ *being an asymmetric ordering, such that*

$$\begin{cases} for\ all\ i \neq k & x_i > x_k \\ for\ all\ i \neq k' & x'_i > x'_{k'} \end{cases}$$

then

$$x_n \overline{P}(\varepsilon) x'_n$$

In all other cases

$$x_n \overline{I}(\varepsilon) x'_n$$

(5) *Drop Separation: similar example as in (5) in the former proof*

■.