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POVERTY MEASUREMENT: SOME FORMAL AND SUBSTANTIVE INTERCONNECTIONS AND A NEW CLASS OF POVERTY INDICES

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Poverty Measurement: some Formal and Substantive Interconnections and a New Class of Poverty Indices

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Abstract

We show how different value judgements driving the determination of individual poverty levels has implications on the informational content of the aggregate index in terms of the *four* "I"s of poverty -namely the "classic" *incidence*, *intensity* and *inequality* dimensions, and the *injustice* dimension which we introduce. A particular way to conceive individual losses from poverty leads to a new class of poverty indices taking a rather unusual form. A member of that class has characterizing properties with respect to *relative* poverty measures.

JEL Classification: D31, D63

1. Introduction

Most poverty indices are derived from an additive approach consisting in the -normalized- summation of individuals' levels of deprivation provided by an appropriate individual deprivation.

Focusing on the FGT [Foster *et al.*, 1984] and BF [Bourguignon and Fields, 1997] classes of poverty measures, we illustrate how the way the "fixed" and "variable" welfare losses from poverty are dealt with at the individual level has crucial implications in terms of the informational content of the aggregate index on what Jenkins and Lambert [1997] call the three "I"s of poverty at the aggregate level -namely the *incidence*, *intensity* and *inequality* dimensions of poverty, emphasized by poverty literature since Sen's [1976] seminal work. It is well-known that when the parameter of poverty aversion grows indefinitely only the poorest tends to matter in the FGT index. The above dimensions become virtually irrelevant and the poverty ordering of different income distributions will be based on a different dimension, that is the condition of the poorest. We like to name this dimension as the "injustice" of poverty -hence a fourth "I" of poverty- along Rawls' "theory of justice" entailing a special concern for the least advantaged.

At the individual level, the α-related weighting of normalized income shortfalls from an exogenous poverty line in the FGT class can be closely associated with the concept of prioritarianism. That can be said for the *way* the discriminative attitude *de facto* takes place -we build on Vallentyne [2003]- and the *reason* why it is done in the realm of "absolute" poverty measurement -we build on Parfit [1995].

We derive a new poverty index -then parametrically generalized- from a particular way to amalgamate the "fixed" and "variable" loss from poverty and/or a peculiar way to take into account the different α-induced "degrees of prioritarianism" in the FGT class. Within that class the possibility is open to enjoy either the properties related to continuity or discontinuity at the poverty line -i.e. those of the FGT class and the BF class, respectively.

The individual deprivation function of the derived index for $\gamma = 0$ is proved to posses a characterizing property. A poverty index is *relative* -i.e. satisfies the Scale Invariance Axiom- if and only if its individual deprivation function is a map (or a composition of maps) of that particular way of considering the α-induced "degrees of prioritarianism" in the FGT class.

2. Continuous and discontinuous losses from poverty: the FGT and BF classes of indices

Bourguignon and Fields [1997] identify two distinct aspects in the individual welfare losses from *absolute* poverty.[1] One arises simply because an individual is poor, in the sense that his income level does not allow him to fulfil the "accepted conventions of minimum needs" [Sen, 1979: 291]. The other reflects the consideration that poverty becomes harsher the further the individual's income gets below the poverty line. Building on the FGT class, they propose a class of loss-from-poverty functions presenting a discontinuity of the first kind at the poverty line and continuous elsewhere. The two classes are presented below.

Consider a fixed and finite set *N* of individuals of size $n \in \mathbb{N}$. Let $y = (y_1, y_2, ..., y_n) \in \mathbb{R}_n^{>0}$ be the correspondent vector of incomes arranged in non-decreasing order, where different subscripts to *y* denote different persons in *N*. Take an exogenous poverty line $z \in \mathbb{R}^{>0}$ and define the individuals in the subset $Q = \{1, 2, ..., q\} \subseteq N$ as poor, where y_q is the largest income smaller than *z* -a *weak* definition of the poor is adopted along the definition in Donaldson and Weymark [1986]. All the indices in this paper are to be intended as attaching a poverty value of zero to individuals at or above *z*; the different functional forms shown refer to the poor subset of the income distribution.

The individual loss-from-poverty function in the class of poverty measures developed in Foster *et al.* [1984] is given by

$$P_{\alpha,i} = (G_i)^\alpha,$$

where $G_i = \frac{z - y_i}{z}$ is the normalized income shortfall from the poverty line -or poverty gap- of the *ith* individual and $\alpha \in \mathbb{Z}^{\geq 0}$ can be interpreted as a parameter of poverty aversion.

The individual loss-from-poverty function in the BF class of poverty measures is given by

$$P_{\alpha,\delta,i} = \delta + (G_i)^\alpha,$$

where $\delta \in \mathbb{R}^{>0}$ and α receives the same interpretation as in $P_{\alpha,i}$ but is taken exceeding unity in order to enjoy the larger set of properties associated to strict convexity.

The aggregate indices for the FGT and BF classes are, respectively:

$$P_\alpha = \frac{1}{n}\sum_{i=1}^{q} P_{\alpha,i} \text{ and } P_{\alpha,\delta} = \frac{1}{n}\sum_{i=1}^{q}\left[\delta + P_{\alpha,i}\right] = \delta H + P_\alpha.$$

Fig. 1

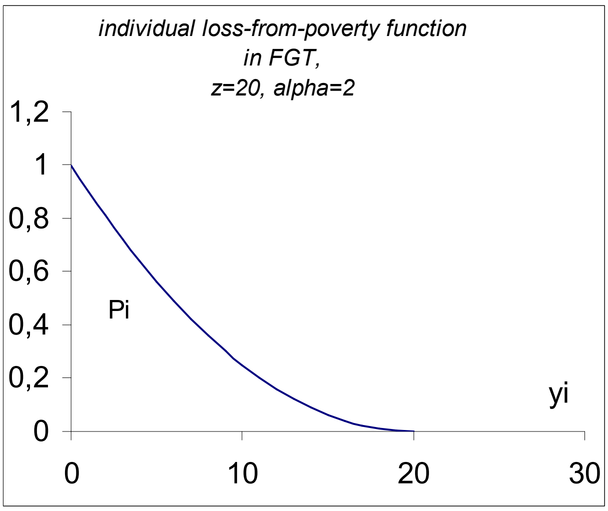
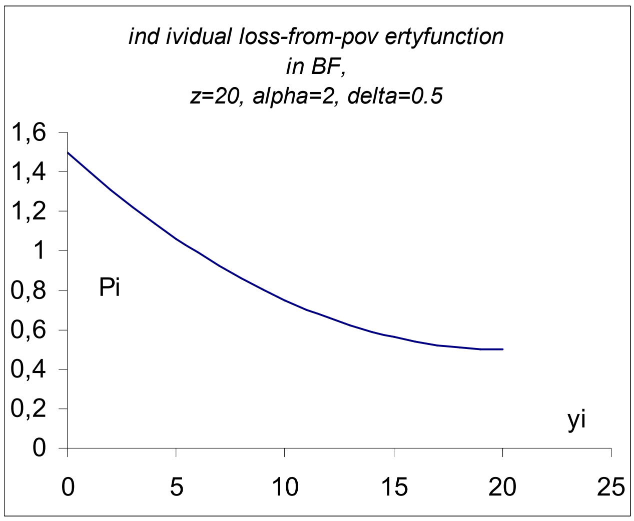


Fig. 2



It is clear from the above figures that the difference between $P_{\alpha,i}$ and $P_{\alpha,\delta,i}$ as multivariable functions of the elements in $\{y_i, z, \alpha\}$ is simply that the latter is an upward translation of the former by the magnitude of a strictly positive real δ. This precisely follows from the motivation in Bourguignon and Fields [1997]. A desirable function should take into account the continuous aspect of the loss from poverty -$P_{\alpha,i}$ is chosen but other functional forms are hinted at- and for each individual a constant should be added by virtue of his condition of being poor. In their words, “a ‘fixed loss’ from poverty which arises in addition to the income-dependent ‘variable loss’ from poverty” (p. 158).

The general justification for continuity is surely quite sensible, in Zheng’s [1997] words: “given a very small change in a poor person’s income, we could not expect a huge jump in the poverty level” (p. 131). Along this argument, continuity at the poverty line is often considered an appealing property for a poverty index; moreover, continuity would diminish the impact of error measurements related to incomes close to z on the overall poverty figure.

A discontinuity at the poverty line is presented by Bourguignon and Fields [1997] as an attempt to integrate the utilitarian framework with the linen-shirt argument/capability approach, which they interpret as bringing about two implications: I1) “an individual too poor to be able to buy the linen shirt suffers shame *absolutely*, i.e. either he has the means to buy the linen shirt… or else he lacks [them]… and suffers shame” (p. 157, emphasis added); I2) “the shame he suffers is discrete -he suffers a full amount of shame even if he is only epsilon short of being able to buy the shirt” (p. 157).

These arguments openly contrast with the influential position of Watts [1968], for whom “poverty is not really a discrete condition. One does not immediately acquire or shed the afflictions we associate with the notion of poverty by crossing any particular poverty line” (p. 325). Interestingly, Donaldson and Weymark [1986] do argue that the practical difficulties in measuring incomes make it reasonable to require continuity, but acknowledge that “On the other hand, the use of a poverty line to sharply demarcate the rich from the poor suggests, but does not require, that a poverty index might be discontinuous at the poverty line” (p. 674). Atkinson [1987] considers a discontinuous index as the headcount acceptable along the interpretation of a minimum income as a basic right, in which case the value of the index would correspond to the number of people deprived of that right. Nevertheless, he recognizes that “there is room for difference of opinion” (p. 754).

In terms of properties, Bourguignon and Fields [1997] see their class as retaining “all the axioms and properties of the P_α index, while also combining with them the insight reflected in the headcount ration on the loss from being poor" (p. 156). More precisely, the BF class satisfies the axioms enjoyed by the distribution-sensitive members of $P_{\alpha}{}^{2}$ but with *Restricted Continuity Axiom* (RCA) and *Weak Transfer Axiom* (WTA) in lieu of *Continuity Axiom* (CA) and *Transfer Axiom* (TA), respectively, where i) RCA indicates the requirement that a poverty measure should be left continuous at the poverty line -and, trivially, also right continuous if it is focussed below the poverty line; ii) the more demanding CA requires continuity in the whole domain; iii) TA prescribes that *any* regressive transfer among poor individuals increases the poverty index; iv) WTA is TA restricted to regressive transfers where the recipient remains poor after the transfer.

The above brief discussion suggests how different views may regard the satisfaction of the pair RCA/WTA rather than the pair CA/TA as a gain or a loss.

3. The FGT and BF classes and the 'three "I"s of poverty'

Sen [1976] is motivated by the development a poverty index that is *adequately informative* on the situation of the poor. The information considered relevant have been labelled as the "three 'I's of poverty", namely the *incidence*, *intensity* and *inequality* dimensions of aggregate poverty.

Sen does value the informative content inherent in the headcount ratio (H) and in the income-gap ratio (I), and asserts that "Both should have some role in the index of poverty" [Sen, 1976: 223]. The proportion of poor individuals in a society provides information about the *incidence* of poverty. The extent to which poor incomes fall short from the poverty line gives indications about the *intensity* of poverty. However, except in the case of a perfectly egalitarian income distribution below the poverty line -Axiom N in Sen [1976]- the use of H and I *alone* is challenged on the ground of their "crudeness". The blame is on the silence about how incomes -or, equivalently, the shortfalls- are distributed among the poor. Such considerations motivated him to develop a "composite measure P" which could also "take note of the *inequality* among the poor" where "G [the Gini coefficient of the poor] provides this information" (p. 227, emphasis added). Various distribution-sensitive indices have been subsequently proposed in the literature, replacing the rank-order weighting used by Sen with other ways to take into account the inequality below the poverty line.

In order to investigate the informational content of the BF class with respect to the three "I"s of poverty, it is helpful to remember that $P_{\alpha=0}=H$, $P_{\alpha=1}=HI$ and the composite nature of $P_{\alpha=2}$. Calling C_p^2 the coefficient of variation of poor incomes, Foster *et al.* [1984] observe that "indeed [$P_{\alpha=2}$] may be expressed as a combination of this inequality measure [C_p^2], the headcount ratio and the income-gap ratio in a fashion similar to Sen [1976]" (p. 761). They show that $P_{\alpha=2}$ can be written as $H\left[I^2+(1-I)^2 C_p^2\right]$ - $P_{\alpha=2}$ increasing in C_p^2. *Mutatis mutandis*, Sen's interpretation of his index in terms of the informational content on the three "I"s of poverty is closely applicable to $P_{\alpha=2}$. Number of poor and poverty deficit being equal between two income distributions, that with the larger inequality will be signalled as having more poverty.

It is straightforward to see that only through the consideration of the income-dependent variable loss from poverty at the individual level it is possible to derive information on aggregate dimensions such as the magnitude and the distribution of the shortfalls -i.e. the intensity and inequality dimensions. It follows that the augmentation of $P_{\alpha,i}$ by a fixed loss from poverty at the individual level operated by the BF class can be seen as inducing at the aggregate level an increase of the relative importance of the incidence dimension of poverty over the intensity and inequality.

We now briefly show how different ways in which the fixed and variable components are melted at the individual level affect the informational content of the aggregate index on the three "I"s of poverty. For mathematical convenience, let us for the moment consider the case in which the magnitude of the 'fixed loss' is chosen to equal unity -the essence of what follows does not change for different values. We can rewrite the individual function and the aggregate measure in the BF class as, respectively:

$$P_{\alpha,i}^{BF}=1+(G_i)^{\alpha^*}=(G_i)^0+(G_i)^{\alpha^*}=\sum_{\alpha=\{0,\alpha^*\}}P_{\alpha,i}$$

and

$$P_{\alpha}^{BF} = \frac{1}{n}\sum_{i=1}^{q}\sum_{\alpha=\{0,\alpha^*\}} P_{\alpha,i} = H + P_{\alpha^*},$$

where the choice of α^* will depend on the degree of aversion to poverty the 'variable loss' is required to exhibit.

Bourguignon and Fields [1997] directly consider values of α^* exceeding unity because they are interested in obtaining a distribution-sensitive measure. Nevertheless, the motivation of amalgamating continuous and discontinuous losses from poverty does not necessarily require the use of a distribution-sensitive term. Researcher R_1 may in fact be well satisfied with expressing the variable loss by the linearity inherent in $\alpha^* = 1$. In that case, the aggregate measure $\tilde{P}_{\alpha}^{BF} = \frac{1}{n}\sum_{i=1}^{q}\sum_{\alpha=\{0,1\}} P_{\alpha,i} = H + P_{\alpha=1} = H + HI$ will be informative on the incidence and intensity but not on the inequality dimension of poverty. Following the interpretation of H suggested by Atkinson [1987], $P_{1,\delta}$ may be seen as providing a figure represented by the proportion of people deprived of the essential right of a minimum income "adjusted" by the normalized poverty deficit -the latter conceivable as an indicator of the per capita amount of resources necessary to lift every poor person out of poverty, aspect particularly valued in Anand [1977] .

Researcher R_2 may appreciate the weights given by $P_{1,\delta}$ to H and I but may be interested in a distribution-sensitive index informative also on the inequality dimension of poverty. He may simply "adjust" $\tilde{P}_{\alpha}^{BF}$ by $P_{\alpha>1}$ and obtain the aggregate measure $\ddot{P}_{\alpha}^{BF} = \frac{1}{n}\sum_{i=1}^{q}\sum_{\alpha=\{0,1,\alpha>1\}} P_{\alpha,i} = H + P_{\alpha=1} + P_{\alpha>1}$, a sort of BF class where the intensity of poverty receives more weight. Researcher R_3 interested in additional transfer-sensitivity properties may "adjust" $\ddot{P}_{\alpha}^{BF}$ -or also $\tilde{P}_{\alpha}^{BF}$ - by $P_{\alpha>2}$.

In this framework the progressive inclusion of terms with larger α will increasingly have the nature of "adjustments", along the pattern described by an exponential function with a positive argument smaller than unity.

Alternative weighting schemes are easily obtainable not only by choosing $\delta \neq 1$ but also via elementary algebraic manipulations. For example, in the index $P^{BF}_{\alpha,k} = \frac{1}{n}\sum_{i=1}^{q}\sum_{\alpha=\{0,1,2,3\}} k^{\alpha} P_{\alpha,i} = H + kP_{\alpha=1} + k^2 P_{\alpha=2} + k^3 P_{\alpha=3}$ the relative importance of distribution-sensitive terms may be increased at pleasure by choosing an arbitrarily large *k>0*.

4. Prioritarianisms, and leximin vs maximin among the poor

$P_{\alpha,\delta,i}$ is mainly conceived to reflect the twofold character of *i*'s social welfare loss from poverty: a fixed and a variable component. The actual choice of $\alpha = 2,3,...$ is a lesser concern with this respect. Yet, that choice has meaningful implications in the way the income-dependent variable loss is indeed asked to depend on income. A relevant implication has to do with a discriminative attitude towards poor individuals with different income levels, associable with the more general principle of vertical equity, "calling for an appropriate differentiation among unequals" [Musgrave, 1990: 113].[3]

The poorer the individual the larger the value of G_i, so that $G_i > G_{i+1} > ... > G_q > 0$ $\forall i = 1,...,q$ and $\max\{G_i\} = G_1$ -i.e. the normalized income gap of the poorest individual. In P_α, for $\alpha = 1$ every income gap receives equal weight; when α exceeds unity, poorer individuals are assigned larger weights *relative* to less poor individuals -the weight being one's own normalized income gap raised to the power $(\alpha - 1)$.

Vallentyne [2003] defines as "prioritarianism" a notion of social justice where the goal of improving people's life is combined with a "special concern" for the worse-off individuals. He states that "Leximin gives, in effect, infinitely greater weight to a worse off person… [whereas] another form of prioritarianism, finitely weighted prioritarianism, gives only *finitely* more weight" (p. 9).

The choice of *a* member of the subclass $P_{\alpha \geq 1}$ can be associated with the concept of *prioritarianism*. Firstly because the weighting scheme described by Vallentyne is *de facto* the one implemented by P_α. Secondly, for the *reason* motivating such weighting scheme in the realm of measures of *absolute*[4] poverty, very closely reflecting the peculiar feature of prioritarianism as expressed in the seminal work [Parfit, 1995]: "only because these people are at a lower *absolute* level. It is irrelevant that these people are worse off *than others*" (p. 23).

The choice of *which* member of the subclass $P_{\alpha>1}$ will reflect the extent to which we want worse off individuals to be "prioritized". Finite values of α induce forms of finitely weighted prioritarianism, whereas for a very atypical "member" - $P_{\alpha\to\infty}$ - the kind of prioritarianism involved is *leximin*. When α increases indefinitely, P_α's weighting behaviour follows precisely Vallentyne's identification of leximin weighting. In fact, between two individuals $P_{\alpha\to\infty}$ gives "infinitely greater weight to a worse off person"; and within the whole population, to the worst off person. It is straightforward to show that whenever *i<j* not only $\lim_{\alpha\to\infty}(G_i)^\alpha \big/ (G_j)^\alpha = \infty$ but also $\lim_{\alpha\to\infty}(G_i)^\alpha \Big/ \left[(G_j)^\alpha + (G_{j+1})^\alpha + ... + (G_q)^\alpha\right] = \infty$.[5] For $\alpha \to \infty$ the income recipients are ranked according to a *lexicographic ordering* and P_α "approaches a 'Rawlsian' measure which considers only the position of the poorest" [Foster *et al.*, 1984: 763].

It seems worth to mention a fundamental difference between *leximin* and *maximin* stressed by Vallentyne [2003], concepts often used interchangeably in the literature on poverty measurement. While the latter gives absolutely no importance to the second worst off, the former requires that: 1) the situation of the worst off should be enhanced as much as possible; 2) to the extent that the implementation of 1) allows, the situation of the second worst off should be enhanced as much as possible, and so on with the third, fourth, etc. worst off persons. In this light, in the realm of P_α we can have maximin only if we consider comparisons -entailing either individuals or whole groups- between poor on the one side and nonpoor on the other side. This holds for α growing indefinitely as well as for whatever other value of the parameter, since the weight to poor persons is *always* larger than zero -however small it may be- while the weight to nonpoor persons *equals* zero. Evidently, this derives from P_α being a focused measure. Therefore, there would be no maximin-like relationships in the concern expressed by P_α for poor individuals having different incomes, but, as shown above, a leximin ordering when $\alpha \to \infty$.

5. A fourth "I" of poverty

As noted above, for larger α the weighting scheme inherent in the FGT class increases the importance of worse-off individuals *relative* to those who are better off. For example, the importance of the third worst-off individual increases relative to that of individuals *4,…,q* but decreases relative to that of the second worst off and the worst off individuals. Only the worst off individual sees his own importance to increase relative to that of all other poor.

A further implication arises if we consider that, by virtue of the additive character, the aggregate -average- poverty value provided by FGT class derives from the sum of individual contributions. If we focus on relative individual contributions to the aggregate poverty value -i.e. the fraction $P_{\alpha,i} / P_{\alpha}$ - the only individual whose share is always monotonically increasing in α is the worst off, however income is distributed. Different values of the parameter α will therefore determine to what extent aggregate poverty is represented by the situation of the worst-off individual. For larger α, the poverty ordering of different income distributions will increasingly be based on a dimension of poverty different from the three "I"s of poverty and consisting in the condition of the worst off -the direct comparability of the normalized income gap across diverse contexts is well known. From the typical Rawlsian conceptualization of justice as a focus on the "least advantaged", we like to name this dimension as the *injustice* dimension of poverty and consider it as a fourth "I" of poverty.[6]

For α growing indefinitely, the poverty ordering of different income distributions tends to be independent of the three "I"s of poverty and exclusively dependent on what we have called the fourth "I" of poverty. Vallentyne [2004] describes "leximin poverty gap" as considering "that there is at least as much poverty in one distribution as in another if and only if the *largest* poverty gap in the first is at least as great as in the second, and so on" [p. 12].

The limitations of such an ordering are evident, since so many relevant aspects of poverty are neglected. Nevertheless, considering this dimension *together with* the other dimensions may be of interest, especially along a more authentically "Rawlsian" approach. Atkinson [1987], Vallentyne

[2000] and Tungodden and Vallentyne [2006] emphasise the misidentification, especially by economists, of the subset of society Rawls addresses his difference principle to. While the "least advantaged" is generally intended as strictly the worst off individual in society, what the philosopher really refers to is the least advantaged *group*, whose benefits should be considered in an aggregative way -i.e. average or total, [Tungodden and Vallentyne, 2006]. Once an appropriate cut-off function identifying the least advantaged group is set, the latter may turn out to be relatively large. Information on its condition may consequently gain a certain interest.

6. A new class of poverty measures

A class of poverty measures with appealing properties can be derived from a particular way of amalgamating the fixed and variable loss from poverty at the individual level. Indefinitely augmenting what we called the extended BF class, the individual deprivation function of the new measure -call it P_i^∞- is the sum of the infinitely many addenda $P_{\alpha,i}$ for $\alpha = 0,1,2,...$ Defined $P_i^M = \sum_{\alpha=0}^{M} P_{\alpha,i} = 1 + P_{1,i} + P_{2,i} + ... + P_{M,i}$ with $M \in \mathbb{Z}^{\geq 0}$, then P_i^∞ is obtained as follows:

$$P_i^\infty = \lim_{M\to\infty} P_i^M = \lim_{M\to\infty} \sum_{\alpha=0}^{M} P_{\alpha,i} = \lim_{M\to\infty} \left(1 + P_{1,i} + P_{2,i} + ... + P_{M,i}\right) = 1 + P_{1,i} + P_{2,i} + ...$$

Recalling that $P_{\alpha,i} = \left(G_i\right)^\alpha$, we are able to recognise a geometric series. Noting that $0 < \left(g_i/z\right) < 1$, we realise that the series is convergent and its value is simply

$$P_i^\infty = \frac{1}{1-(G_i)} = \frac{z}{y_i}. \tag{1}$$

The aggregate index is given by

$$P^\infty = \frac{1}{n}\sum_{i=1}^{q}\frac{z}{y_i}.$$

The index is obtained by including *ad infinitum* all the possible degrees of prioritarianism described by the functional form of $P_{\alpha,i}$. By virtue of the progressively smaller magnitude of each addendum as $\alpha \to \infty$, the poverty value can be thought as the result of a process of "adjustment" by stronger forms of prioritarianism. The idiosyncratic values $P_{i=1}^{\infty}, P_{i=2}^{\infty}, ..., P_{i=q}^{\infty}$ can indeed be thought as capturing a genuine "Rawlsian" element, in that "adjusted" by components $l_{i=1}, l_{i=2}, ..., l_{i=q}$ which are ranked in a *pure* lexicographic ordering -though infinitesimal and clearly of negligible empirical relevance. It is straightforward to verify that P^{∞} satisfies all the poverty axioms met by the BF class.

By writing $P_i^{\infty} = 1 + \tilde{P}_i^{\infty}$, our measure can be strictly associated to the BF class where the "fixed loss" from poverty is given by $\delta = 1$ and the income-dependent 'variable loss' from poverty is given by

$$\tilde{P}_i^{\infty} = \lim_{M \to \infty} \tilde{P}_i^{M} = \lim_{M \to \infty} \sum_{\alpha=1}^{M} P_{\alpha,i} = \lim_{M \to \infty} \left(P_{1,i} + P_{2,i} + ... + P_{M,i} \right) = P_{1,i} + P_{2,i} + ...,$$

where for the above reasons, we derive

$$\tilde{P}_i^{\infty} = \frac{G_i}{1 - G_i} = \frac{z}{y_i} - 1. \qquad (2)$$

Fig. 3

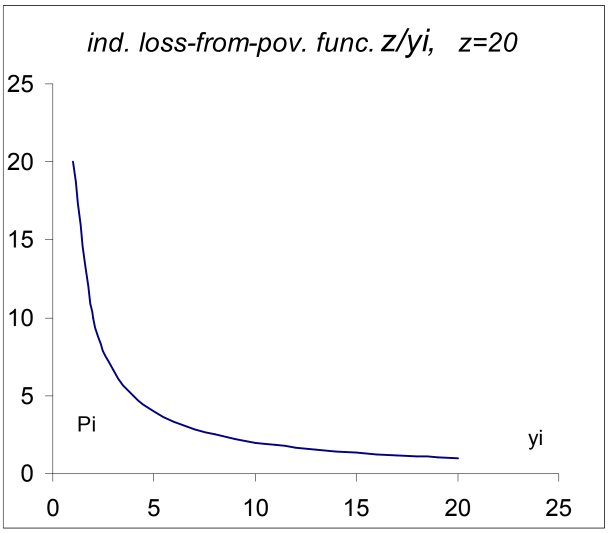
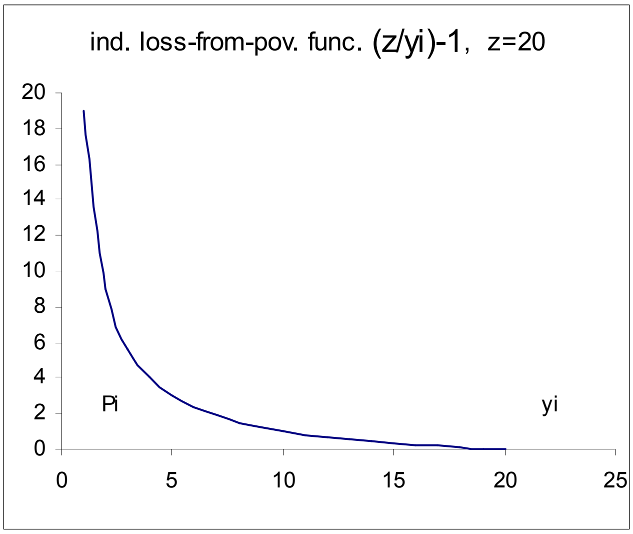


Fig. 4



Evidently, $\tilde{P}_i^{\infty}$ represents a downward translation by the magnitude of $\delta = 1$ of P_i^{∞} -the researcher can of course choose a $\delta^* \neq 1$ and adopt a measure like $P_i^{*\infty} = \delta^* + \tilde{P}_i^{\infty}$.

$\tilde{P}_i^{\infty}$ alone may be preferred to P_i^{∞} if one rejects the notion of a "fixed loss" from poverty at the individual level and deems the stronger forms of continuity and transfer axioms compelling. $\tilde{P}^{\infty}$, the aggregate measure built upon $\tilde{P}_i^{\infty}$, shares in fact the FGT class' set of axioms while P^{∞} shares the BF class' ones.

P_i^{∞} and $\tilde{P}_i^{\infty}$ are not bounded above by unity, like the individual loss-from-poverty function in Watts [1968] but differently from most measures proposed in poverty literature. We agree with Zheng [1993], who notes that "there is no strong reason for limiting the index to be no greater than one" (p. 84).

It is possible to generalize the above indices in a parametrically defined class by applying the general algebra of geometric series. The individual deprivation function and the aggregate measure are expressed as follows, respectively:

$$P_{\gamma,i}^{\infty} = \lim_{M\to\infty}\sum_{\alpha=\gamma}^{M} P_{\alpha,i} = \frac{(G_i)^{\gamma}}{1-G_i} = \frac{z}{y_i}(G_i)^{\gamma}, \tag{3}$$

and

$$P_{\gamma}^{\infty} = \frac{1}{n}\sum_{i=1}^{q} P_{\gamma,i}^{\infty},$$

where the choice of the poverty-aversion parameter $\gamma \in \mathbb{Z}^{\geq 0}$ has implications at both and the aggregate and the individual level. In particular:

At the individual level, γ indicates the "softest" degree of prioritarianism -among those described by $P_{\alpha,i}$ - included in the measure. If $\gamma \geq 1$, only the variable loss from poverty is taken into account and the index will enjoy continuity-related properties; if $\gamma = 0$ the individual deprivation function is discontinuous reflecting the belief in the existence of a fixed loss from poverty alongside the variable loss. For $\gamma = 0$, the individual deprivation function -which we named P_i^{∞}- possesses a unique property, as shown in the following section. The individual deprivation functions for $\gamma = 0$ and $\gamma = 1$ represent, respectively, the poverty line and the poverty gap expressed as proportions of i's income.

At the aggregate level, in terms of informational content on the poverty dimensions, as noted above the larger γ the larger the relative weight to the fourth "I" of poverty; for γ growing indefinitely, only the "injustice" of poverty tends to matter, intended as the condition of the poorest. The member of the P_{γ}^{∞} class most informative on the "incidence" of poverty is $P_{\gamma=0}^{\infty}$ and the one most informative on the "intensity" of poverty is $P_{\gamma=1}^{\infty}$.

Moreover, a further straightforward interpretation of the P_{γ}^{∞} class can be offered. Noting that $P_{\gamma,i}^{\infty}=\frac{(G_i)^{\gamma}}{1-G_i}=\frac{z}{y_i}(G_i)^{\gamma}$, it is easy to see that each member of that class can be simply thought of as the correspondent -i.e. the values of γ and α being the same- member of the P_{α} class weighted by the term $\frac{z}{y_i}$.

Through the value assigned to the parameters α and γ, respectively, with the P_{α} class we chose one degree of prioritarianism while with the P_{γ}^{∞} class we chose "from where" including different degrees. Another possibility, implicitly dealt in Section 3 is the choice of the strongest degree of prioritarianism to adjust for.

The three options are accommodated by the following formulation:

$$P=P_{\gamma,i}^{\infty}-P_{\beta,i}^{\infty}, \tag{4}$$

where $P_{\beta,i}^{\infty}=\lim_{M\to\infty}\sum_{\alpha=\beta}^{M}P_{\alpha,i}=\frac{(G_i)^{\beta}}{1-G_i}$ and $\beta>\gamma$.

As shown in Table 1, when $\beta=\gamma+1$ the members of the P_{α} class are generated, and when $\beta\to\infty$ are those of the P_{γ}^{∞} class. When $\beta=\gamma+h, h\in\mathbb{Z}, 1<h<\infty$, P is a sum of finitely many addenda each represented by a different power of the normalized poverty gap –i.e. by a different member of the P_{α} class.

Table 1: Different combinations of parameters β and γ, $\beta > \gamma$.

β / γ	0	1	2	3	4	∞
0	X	$P_{\alpha=0}$	$P_{\alpha=0}+P_{\alpha=1}$	$P_{\alpha=0}+P_{\alpha=1}+P_{\alpha=2}$	$P_{\alpha=0}+P_{\alpha=1}+P_{\alpha=2}+P_{\alpha=3}$	$P^{\infty}_{\gamma=0}$
1	X	X	$P_{\alpha=1}$	$P_{\alpha=1}+P_{\alpha=2}$	$P_{\alpha=1}+P_{\alpha=2}+P_{\alpha=3}$	$P^{\infty}_{\gamma=1}$
2	X	X	X	$P_{\alpha=2}$	$P_{\alpha=2}+P_{\alpha=3}$	$P^{\infty}_{\gamma=2}$
3	X	X	X	X	$P_{\alpha=3}$	$P^{\infty}_{\gamma=3}$
4	X	X	X	X	X	$P^{\infty}_{\gamma=4}$
∞	X	X	X	X	X	X

7. P_i^{∞} and relative poverty measures

Blackorby and Donaldson [1980] define a poverty measure as *absolute* or *relative* if, respectively, the addition or the multiplication by a -positive- scalar of all incomes and the poverty line leaves the value of the index unchanged. Absolute indices are said to satisfy *Translation Invariance Axiom* and relative indices *Scale Invariance Axiom*. As shown by Zheng [1994], for distribution-sensitive poverty indices these are mutually exclusive properties and only the class of headcount-related poverty indices[7] can be both absolute and relative. We will call this class as *absolute-relative* indices. The above taxonomy is not exhaustive, since a poverty index can be neither absolute nor relative -e.g. the index in Hagenaars [1987].

Following Foster and Shorrocks [1991: 701] and Zheng [1993: 85], necessary and sufficient condition for *Scale Invariance* -taken without proof- is the existence of a function $\varphi(\cdot)$ such that $P_i(y_i, z)$ can be written as $\varphi(\frac{z}{y_i})$. Recalling that $P_i^\infty = \frac{z}{y_i}$, we state the following and provide a proof.

Proposition. *Let* $(y', z') = \lambda(y, z)$, $\lambda > 0$. *Then* $P(y,z) = P(y', z') \Leftrightarrow P_i(y_i, z) = \varphi(P_i^\infty)$. *In other words, a poverty index* $P(y,z)$ *is relative if and only if its individual loss-from-poverty function is a transformation of* P_i^∞.

Proof. The sufficiency side of the proposition is obvious. The necessity side states that $P_i(\lambda y_i, \lambda z) = P_i(y_i, z) \Rightarrow \exists \varphi : P_i(y_i, z) = \varphi(P_i^\infty)$. We express $P_i(y_i, z)$ in polar coordinates as $\breve{P}(r, \theta)$, our hypothesis becoming $\breve{P}(r,\theta) = \breve{P}(\lambda r, \theta)$. $\forall r'$ we can write $\breve{P}(r', \theta) = \breve{P}(\frac{r'}{r} r, \theta) = \breve{P}(\lambda r, \theta)$ which equals $\breve{P}(r,\theta)$ by hypothesis. $\breve{P}(r,\theta)$ can be therefore written as $\hat{P}(\theta)$. Reminding that $\theta = arctg\frac{z}{y_i}$, we have $\hat{P}(\theta) = \hat{P}(arctg\frac{z}{y_i}) = \varphi(\frac{z}{y_i}) = \varphi(P_i^\infty)$.

Q. E. D.

The above proposition *characterizes* all relative poverty measures as functions of P_i^∞. All relative poverty indices can be seen as a function -or composition of functions- of the sum *ad infinitum* of all the members of the FGT class. It will be the transformation inherent in $\varphi(\cdot)$ to establish the functional form of a particular individual deprivation function, as well as the concern expressed for the "four 'I's of poverty".

For example, if the relative poverty index is represented by the very FGT class, then $P_i(y_i, z) = P_{i,\alpha}$ and we can write:

$$P_{i,\alpha} = (1 - \frac{y_i}{z})^\alpha = \varphi(P_i^\infty),$$

where $\varphi(\cdot) = h\{g[f(\cdot)]\}$ with $f(\cdot) = \left(P_i^\infty\right)^{-1}$, $g(\cdot) = 1 - f(\cdot)$ and $h(\cdot) = [g(\cdot)]^\alpha$. Different values of α will offer the precise functional form of $h(\cdot)$, with the well-known implications in terms of functional form, properties enjoyed and concern for the poverty dimensions.

Somewhat trivially, one may say that when $\varphi(\cdot)$ is the so-called *identity function* then $P_i(y_i, z)$ is P_i^∞ itself. If instead $\varphi(\cdot)$ is the *logarithmic function*, then $P_i(y_i, z)$ becomes the individual loss-from-poverty function in Watts [1968], call it P_i^W :

$$P_i^W = \log z - \log y_i = \log \frac{z}{y_i} = \log P_i^\infty .$$

The individual deprivation function in the Watts index can be seen as the logarithm of the geometric series represented by the sum of all the members of the P_α class. Similarly to what we saw comparing P^∞ with $P_{\alpha>1}$ -or also with $\tilde{P}^\infty$, the difference between the properties enjoyed by P^∞ and the Watts index regards the satisfaction of RCA and WTA in lieu of CA and TA, respectively.

Conclusions

Different ways to conceive the welfare losses from poverty at the individual level have implications on the informational content of the aggregate index on what can be called the four "I"s of poverty at the aggregate level. We show various interconnections leading to an atypical derivation of a class of poverty measures and a characterization of relative poverty indices. Further research is necessary for a more complete and precise conceptualization of the analytical results obtained.

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[1] With the term "absolute" we simply indicate that we refer to poverty as shortage from a poverty line that is independent on others' income.

[2] Like the Chakravarty [1983] index, the Clark *et al.* [1981] second family of indices and the Hagenaars [1987] index, $P_{\alpha>1}$ satisfies *Symmetry*, *Focus*, *Replication Invariance*, *Monotonicity*, *Transfer*, *Transfer Sensitivity*, *Subgroup Consistency*, *Decomposability*, *Continuity*, *Normalization* and *Scale Invariance*.

[3] If δ is considered as given, an implication of different nature derives straight from the exponential behaviour. For the *ith* individual the choice of larger α comes along with the decrement of the importance of the variable loss relative to the fixed loss; also, individual poverty values would be getting more and more similar to each other and getting closer to δ. However, even in the case of an exogenous δ elementary algebraic manipulations similar to the one shown at the end of the previous section allow for weighting schemes strengthening at pleasure the relative importance of the idiosyncratic variable loss as long as α is kept finite.

[4] We use this term as opposed to *relative* or *social* poverty, in order to identify poverty as the inability to fulfil a certain set of minimum physical needs -i.e. food, clothes, shelter, etc. In the assessment of absolute poverty, i's "suffering" is exogenous from other's income levels.

[5] By multiplying numerator and denominator by $1/(G_j)^{\alpha}$ the result becomes evident.

[6] Atkinson [1987] notes how the difference principle has nothing to do with poverty *per se* and poverty would more naturally enter Rawls' theoretical framework through his first principle. In fact, the argument in the difference principle is an ordinal one and the least advantaged may be well above the poverty line; instead, the first principle postulates priority to be given to the basic liberties, a necessary condition for which can be identified in a minimum income level. However, we may simply think that whenever the set of poor individuals in society is not an empty set, then the difference principle can be of interest in a discussion on poverty in that the least advantaged is surely below the poverty line.

[7] Defined by Zheng as the measures which can be expressed as a function of the number of the poor and the population size.