Società italiana di economia pubblica
XV CONFERENZA

DIRITTI, REGOLE, MERCATO
Economia pubblica ed analisi economica del diritto

Pavia, Università, 3 - 4 ottobre 2003

BILATERAL OLIGOPOLY: A PRELUDE

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XV Conferenza SIEP - Pavia, Università, 3 - 4 ottobre 2003

pubblicazione internet realizzata con contributo della COMPAGNIA di San Paolo

società italiana di economia pubblica

dipartimento di economia pubblica e territoriale – università di Pavia

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Very Preliminary Version

1 Introduction

The aim of this paper is to provide a preliminary approach to the analysis of the process of endogenous price formation in thin markets. In such markets, also denoted as bilateral oligopolies, both sides, being typically concentrated, have some market power, so that both buyers and sellers are able to affect the prices at which they trade.

Examples of bilateral oligopolies may be found in basic commodities markets - such as the ones for the coffee, tobacco or minerals - in the energy markets and in the intermediate goods markets - such as most the manufacturing industries, the aerospace or defence industries, the hi-tech.

As a few pioneering studies have recently pointed out (Bjornerstedt and Stennek (2001), Inderst and Wey (2003)), the process of price formation in bilateral oligopolies is rather peculiar. Indeed, it is very unlikely that the traders on any side of the market may behave as price-takers. Rather, it seems reasonable to think to the formation of the price as the outcome of a complex of negotiations among traders. The mentioned studies have argued that bilateral oligopolies may be reduced to a simple collection of many bilateral monopolies: the prices, thus, may emerge as the outcome of many simultaneous Nash-bargaining cooperative solutions, each involving an exogenously matched pair of one seller and one buyer.

In this paper, at the contrary, we focus on non-cooperative bargaining solutions, and we attempt to extend one of the most common models from the literature to a bilateral oligopoly where all the sellers and the buyers can simultaneously negotiate while not being constrained by a fixed partner.

In the literature on non-cooperative bargaining in decentralized markets it is traditionally assumed that buyers and sellers are pairwise matched through some random procedure, and that the order in which agents can make or respond to price offers is exogenously given.

However, as Chatterjee and Dutta (1998) observe, while these assumptions are acceptable when modelling large anonymous markets, they are less appropriate in thin markets where the search costs are usually low, and, particularly when agents are heterogeneous, traders may have interest in choosing their partner.

Chatterjee and Dutta have provided a first insight into the effect of competition for bargaining partners on the price - or, the prices - that prevail in thin markets, as well as how the matches themselves are simultaneously determined. However, they have focused only on alternating offers negotiations and on the special cases of targeted and "*telephone-calls*" bargaining procedures whose results can not be convincingly extended to other procedures.

At the contrary, we focus on a model of negotiations with public offers and a random order of proposers, which has been usually adopted for the analysis of bargaining in large decentralized markets (see for instance Rubinstein and Wolinsky (1990), Gale (1986), and, specially, De Fraja and Sakovics (2001)) for being easily comparable with the outcome of a Walrasian competitive market.

The aim of this preliminary analysis is to explore the strategic non-cooperative micro-foundations of price formation in markets with a limited number of traders, along the way already investigated for the case of large decentralized markets.

2 The Model

We focus on the simplest case of a bilateral duopoly, where two identical sellers, S_1 and S_2 each own one single unit of an indivisible good. Both sellers have the same reservation value of zero for the good.

There are also two heterogeenous buyers, B_1 and B_2, both of whom demand one unit each of the commodity. The buyers' valuations are $v_1 = 1$ and $v_2 = \lambda$, respectively, with $1 > \lambda > \frac{1}{2} > 0$. All the valuations are common knowledge.

The prices at which the good is exchanged if trade takes place is exclusively determined by endogenous bargaining among the players. In particular, we assume that the all the traders in the thin market negotiate according to a public offers bargaining procedure with random order of proposers.

In each period $t \in \{1, 2, ...\}$, one side of the thin market is randomly selected to propose offers: both the supply and the demand sides may be selected to be the proposers with equal probability $\frac{1}{2}$, independently by the past histories and random draws.

The agents on the side of the market which has been selected - for instance the buyers B_i with $i = 1, 2$ - each simultaneously announce a price p_i at which they are willing to buy one unit of the good.

The two sellers then respond, again simultaneously, to the price offers. A response is either acceptance of *one* offer or rejection of both offers.

If *both* sellers accepts B_i's offer of p_i, then the two sellers are matched with equal probability with B_i. At the contrary, both pairs are matched if the two sellers accept offers from different buyers.

Matched pairs leave the market with the good being exchanged at the agreed price offer. If some pairs remain unmatched at the end of period t, then in period $t + 1$ the game is repeated with sellers making price offers and buyers responding to these offers, agents on both sides of the market moving simultaneously as in period t. This procedure is repeated so long as some pair remains in the market.

All agents have a common discount rate $\delta \in (0, 1)$. Thus, if one unit of the good is exchanged in period t between B_i and S_j at the price p, then the payoff of $B_i = \delta^{t-1} (v_i - p)$ and the payoff of the seller $S_j = \delta^{t-1} p$.

First, note that the equilibrium cannot entail bargaining forever. The reason is that, if it did, B_1 could deviate and offer a price δv_1, which would be accepted by either seller, thus giving a positive payoff to the deviator. If an equilibrium exists therefore, it must consist of two agreements, either both in period t or in periods $(t, t + 1)$.

Also, note that if only one pair reaches agreement in period t, then the remaining pair will be engaged in a Rubinstein alternating offers subgame in the next period.

To select a candidate to the equilibrium may be not trivial in this game. In fact, economic intuition would only vaguely suggest that the two sellers will compete each other in the attempt to sell to the highest-valuation buyer. Co-existence of two different prices can not be a priori ruled out, as well as delays in the trade.

Indeed, the identification of the set of the potential candidates to the equilibrium may be sequentially restricted by the help of a process of sequential elimination of the game's outcomes which are evidently impossible.

As the game is of perfect information, in the following analysis we will focus on solutions in stationary pure-strategies subgame perfect equilibria.

Thus, observe that, by a standard argument by theory of stationary games (see for instance Osbourne and Rubinstein (1990)), a stationary sequential game may be fully described at any time by describing all its possible subgames. In particular, define S-games the subgames of the original game that start when the sellers are randomly selected to make offers; analogously define B-games the subgames of the original game that start when the buyers are randomly selected to make offers. Hence, the analysis of the equilibria in the original sequential game is perfectly equivalent to the investigation of the subgame perfect equilibria in both the S-games and the B-games.

2.1 S-games

Consider all the subgames of the original game starting with the selection of the sellers as proposers. In these subgames we denote p_1 and p_2 the price offered simultaneously and independently by sellers S_1 and S_2 respectively.

We now describe the conditions for all possible subgame perfect equilibria emerging in a S-game. It is easy to check that exactly 9 potential equilibrium allocations of the goods may emerge in a subgame starting in a bargaining period in which the sellers make proposals. In fact, if one equilibrium does exist it must necessarily be one from the following allocations:

$$\begin{array}{cc} S_1 & S_2 \\ p_1 & p_2 \\ \varnothing & \varnothing \end{array} ; \begin{array}{cc} S_1 & S_2 \\ p_1 & p_2 \\ B_1 & \varnothing \end{array} ; \begin{array}{cc} S_1 & S_2 \\ p_1 & p_2 \\ \varnothing & B_1 \end{array} ; \begin{array}{cc} S_1 & S_2 \\ p_1 & p_2 \\ B_2 & \varnothing \end{array} ; \begin{array}{cc} S_1 & S_2 \\ p_1 & p_2 \\ \varnothing & B_2 \end{array}$$

$$\begin{array}{cc} S_1 & S_2 \\ p_1 & p_2 \\ B_1, B_2 & \varnothing \end{array} ; \begin{array}{cc} S_1 & S_2 \\ p_1 & p_2 \\ \varnothing & B_1, B_2 \end{array} ; \begin{array}{cc} S_1 & S_2 \\ p_1 & p_2 \\ B_1 & B_2 \end{array} ; \begin{array}{cc} S_1 & S_2 \\ p_1 & p_2 \\ B_2 & B_1 \end{array},$$

where the last row indicates the set of the buyers accepting the price p_i by the seller S_i, $i = 1, 2$.

We classify the 9 possible allocations in 4 classes and we show that some of them can never represent a subgame perfect equilibrium of the S-games, because of contradictions in the conditions to hold. We thus

gradually restrict the set of the potential equilibria to fewer classes of cases. Finally, by having eliminated all the cases from the set, we show that in the S-games there are no stationary pure-strategies subgame perfect equilibria.

2.1.1 First Class: Both Buyers Reject Both Prices

The first case emerges where both buyers reject both the offers p_1 and p_2 by the two sellers. We may represent the candidate equilibrium by the figure

$$\begin{array}{cc} S_1 & S_2 \\ p_1 & p_2 \\ \varnothing & \varnothing \end{array}.$$

In such a case all the players do not trade and enter the next round, with a new selection of the proposers. Their relative surplus are given by the discounted value of the expected payoffs by entering a new stage of negotiation. It may be checked that this case can never constitute a subgame perfect equilibrium.

First, note that if this was indeed the sellers' equilibrium stationary strategy, must be the case that in any period, when they are selected to make offers, both the sellers keep on proposing prices so high that both buyers reject them.

This means that the offered prices make both buyers worse off than their continuation payoffs. That is, the followings must hold: $p_i > 1 - \delta W(B_1)$ and $p_i > \lambda - \delta W(B_2)$, with $i = 1, 2$ and with $W(B_j)$ being the expected payoff by buyer $j = 1, 2$ by rejecting the offers and entering a new round.

Consider now the lower bound of $W(B_1)$, that is the minimum surplus buyer B_1 may expect from trade by entering a new round of negotiations.

If the one described above is indeed a stationary equilibrium, at every period with probability $\frac{1}{2}$ the sellers' offers are rejected and all the traders go further with the negotiation. Alternatively, again with probability $\frac{1}{2}$, the buyers are selected to make offers.

In such a case, it is easy to show that the minimum payoff buyer B_1 may obtain by proposing a price is $1 - \lambda - \varepsilon$, with ε infinitesimally small. In fact, B_1 can always get at least that surplus by proposing a price $\lambda + \varepsilon$, which, as we will show below, is immediately accepted by both the sellers since it is above the highest possible price offered by B_2. Note, incidentally, that, as $\lambda > \frac{1}{2}$, such a price makes B_1 worse off than any bilateral negotiation with a single seller.

Thus define $W_{\min}(B_1)$ as the lowest continuation payoff buyer B_1 may expect if the above case was indeed a subgame perfect equilibrium: having shown that $W_{\min}(B_1) = \frac{1}{2}\delta W_{\min}(B_1) + \frac{1}{2}(1 - \lambda - \varepsilon)$ gives $W_{\min}(B_1) = \frac{1-\lambda-\varepsilon}{2-\delta}$.

Then, for the above condition $p_i > 1 - \delta W(B_1)$ with $i = 1, 2$ holding in equilibrium, must necessarily be that $p_i > 1 - \delta W_{\min}(B_1) = \frac{1+\lambda+\varepsilon-\delta}{2-\delta}$.

Consider now the sellers. As they propose offers that are rejected by both buyers, their expected payoffs equal $\delta W(S_i)$, where $W(S_i)$ is the expected payoff by seller $i = 1, 2$ by entering a bargaining round before a new selection of the proposer.

Define $W_{\max}(S_i)$ as the highest payoff each seller $i = 1, 2$ may expect from a new bargaining period if their above strategies were indeed a subgame perfect equilibrium. These are necessarily associated to the lowest expected payoffs by the buyers when the latter would be selected to make an offer. That is, the maximum the sellers may obtain by keep on proposing offers that will be rejected by both buyers needs to correspond to a price $\lambda + \varepsilon$ proposed by B_1 when the buyers are selected to make an offer.

In such a case, it may be checked that in a subgame perfect equilibrium both sellers would accept the offered price $\lambda + \varepsilon$, since a rejection will clearly give a lower payoff. In fact, if, say, S_1 rejected the price $\lambda+\varepsilon$, the best it might happen is that S_2 also rejected that price, which gives at most a payoff of $\delta W_{\max}(S_i)$, that is by definition lower than what she would get accepting it. However, if S_2 will accept the price $\lambda+\varepsilon$, S_1 can obtain only $\delta\frac{\lambda}{2}$ in the subsequent bilateral negotiation with B_2.

As both sellers in equilibrium will accept the same price $\lambda+\varepsilon$ by B_1, a random selection of a winner will be in order to solve the tie in the allocation: one of the two seller will be chosen to buy from B_1 at price $\lambda + \varepsilon$, while the other will enter a bilateral negotiation with B_2.

Thus, if their above strategies were indeed a subgame perfect equilibrium, the symmetric highest payoff each seller may expect from a new bargaining round is $W_{\max}(S_i) = \frac{1}{2}\delta W_{\max}(S_i) + \frac{1}{2}\left[\frac{\lambda+\varepsilon}{2} + \frac{\delta}{2}\left(\frac{\lambda}{2}\right)\right]$, that is $W_{\max}(S_i) = \frac{\lambda}{4}\left(\frac{2+\delta}{2-\delta}\right) + \frac{\varepsilon}{2(2-\delta)}$.

Hence, if the sellers adopted the above strategies such that the equilibrium offered prices are never accepted, that is if $p_i > \frac{1+\lambda+\varepsilon-\delta}{2-\delta}$, then they would obtain at most an expected payoff $\delta W_{\max}(S_i) = \delta\left[\frac{\lambda}{4}\left(\frac{2+\delta}{2-\delta}\right) + \frac{\varepsilon}{2(2-\delta)}\right]$. But, then, it is easy to verify that the latter strategies can not be an equilibrium.

In fact, let one of the seller, say S_1, to deviate by proposing, for instance, a price p_1 exactly equal to $\frac{1+\lambda+\varepsilon-\delta}{2-\delta}$. This is the lowest price that leaves the high-valuation buyer indifferent between accepting and rejecting an offer. By comparison, it is immediately checked that this strategy makes S_1 better off with respect to the one of proposing unacceptable offers: in fact, for any value of δ and for small ε, it always holds that $\lambda > \frac{2[\varepsilon(\delta-2)+2(\delta-1)]}{4-2\delta+\delta^2}$, that is, offering p_1 makes S_1's profit strictly bigger than $W_{\max}(S_1)$, the maximum payoff she may expect with the latter strategy. Furthermore, by analogy it may be shown that, even proposing a price so low that it may attract the lowest-valuation buyer, one of the seller may benefit by deviating from the described strategy, at least for large ranges of the relevant primitive parameters. Thus, the described strategies can never be a subgame perfect equilibrium.

2.1.2 Second Class: One Buyer Accepts a Price, the Other Rejects Both Offers

The second possible situation emerges when only one from the two buyers accepts one price, while the other rejects both. This situation includes four cases, depending on the identities of the buyer who accepts and of the seller who proposes the price:

$$\begin{array}{cc} S_1 & S_2 \\ p_1 & p_2 \\ B_1 & \varnothing \end{array} \;;\; \begin{array}{cc} S_1 & S_2 \\ p_1 & p_2 \\ \varnothing & B_1 \end{array} \;;\; \begin{array}{cc} S_1 & S_2 \\ p_1 & p_2 \\ B_2 & \varnothing \end{array} \;;\; \begin{array}{cc} S_1 & S_2 \\ p_1 & p_2 \\ \varnothing & B_2 \end{array} .$$

Given the symmetry in the game related to the existence of two identical sellers, we only consider the allocations as represented by the first and the third figures.

In such cases, only one buyer trade immediately with a seller at the proposed price, while the other buyer enters, in the following period, a bilateral negotiation with the remaining seller. We model the latter negotiation as a Rubinstein bilateral bargaining with random selection of the proposer at every period. Hence, both the remaining buyer and the unmatched seller expect from the bilateral negotiation one-half of the possible surplus to be divided. Thus, in the first case both B_2 and S_2 each expects a discounted payoff of $\delta\frac{\lambda}{2}$, while in the third case, both B_1 and S_2 each expects a discounted payoff of $\delta\frac{1}{2}$.

First note that the third and the fourth cases represented in the figure, in which is the low-valuation buyer to accept one price, intuitively can never be an equilibrium. In fact, it must be always the case that, if a proposed price is accepted by the buyer with the lowest valuation, it should be accepted also by the highest valuation buyer.

The last two cases represented in the figure indeed do not make much sense. However, it may well be the case that in a subgame perfect equilibrium only the high-valuation buyer accepts a price. We now show that neither this case can be a subgame perfect equilibrium. The reported proof refers to the first case in the figure, but it clearly extends by symmetry to the second case, and, a fortiori, to the other two.

Consider the case where, as the outcome of the negotiation, buyer B_1 accepts the price proposed by seller S_1, while buyer B_2 rejects both the prices offered by the two sellers.

The resulting allocation of the goods would be that B_1 buys from S_1 at price p_1, while the low-valuation buyer would trade in a bilateral negotiation with S_2 after some delay. The resulting expected payoffs from such an allocation would be $V(S_1) = p_1$, $V(B_1) = 1 - p_1$, $V(S_2) = V(B_2) = \delta\frac{\lambda}{2}$.

Notice that, if this allocation was a subgame perfect equilibrium, it would be the case that the following conditions were satisfied.

First, it must be the case that $p_2 \geq p_1$, for otherwise B_1 had accepted the lower price p_2 instead.

Second, for the price p_1 to be accepted by buyer B_1 it must be set to a level such that the latter is indifferent between accepting it, gaining $1-p_1$, and rejecting it going to a further bargaining period in a situation such as the one described in the First Class.

Third, must be the case that, by rejecting both offers, buyer B_2 expected an higher payoff than by accepting one of the two. In particular, if buyer B_2 accepted the same price p_1, he would be randomly selected with probability $\frac{1}{2}$ to keep the good rather than going to bilateral negotiations with the remaining seller. Then if this was an equilibrium it must be that the expected payoff for B_2 by accepting p_1 would never be as high as the payoff he may obtain by rejecting and going directly to bilateral negotiation, that is the following must hold: $\frac{1}{2}(\lambda - p_1) + \frac{1}{2}\left(\delta\frac{\lambda}{2}\right) \leq \delta\frac{\lambda}{2}$.

The latter implies that the emerging price would be such that $p_1 \geq \lambda\left(1 - \frac{\delta}{2}\right)$, which in turn also implies, with the first condition, that $p_2 \geq \lambda\left(1 - \frac{\delta}{2}\right)$, the condition that guarantees that buyer B_2 never accepted price p_2.

Having derived these conditions to hold when the above allocation emerges, one may verify that the latter can never constitute a pure-strategies subgame perfect equilibrium.

In fact, consider a deviation by S_2 from the described strategy. For instance, she may deviate by proposing a price $p_2' = p_1 - \varepsilon$, with ε infinitesimally small and, clearly, $p_1 \geq \lambda\left(1 - \frac{\delta}{2}\right)$. We now show that this is indeed a profitable deviation, as S_2 will surely sell the good to buyer B_1, being able to earn $p_1 - \varepsilon$ rather than the lower $\delta\frac{\lambda}{2}$. In fact, the condition $p_1 \geq \lambda\left(1 - \frac{\delta}{2}\right)$ implies that $p_2' = p_1 - \varepsilon \geq \lambda\left(1 - \frac{\delta}{2}\right) - \varepsilon$.

However, as ε is so small that $\varepsilon \leq \lambda(1-\delta)$, that is as $\varepsilon \longrightarrow 0$ as $\delta \longrightarrow 1$, it always holds that $\lambda\left(1 - \frac{\delta}{2}\right) - \varepsilon > \delta\frac{\lambda}{2}$, which in turn implies that, by proposing p_2', S_2 sells the good to buyer B_1 and can profitably deviate from the original situation.

But this contradicts the assumption that the present allocation was an equilibrium. In fact, the described allocation can never constitute a subgame perfect equilibrium, because of the competition among the sellers for capturing the high-valuation buyer's surplus. For the other three cases in the figure, the same logic applies.

2.1.3 Third Class: Both Buyers Accept a Price from the Same Seller

The third possible situation emerges when both buyers accept the price offer from the same seller. This situation includes two perfectly symmetric cases, of which we will only consider the first:

$$\begin{array}{ll} S_1 & S_2 \\ p_1 & p_2 \\ B_1, B_2 & \varnothing \end{array} ; \begin{array}{ll} S_1 & S_2 \\ p_1 & p_2 \\ \varnothing & B_1, B_2 \end{array} .$$

In such a case, one of the two buyers would be randomly selected to trade with S_1, while the other will be matched in the next period with the remaining seller, starting a bilateral negotiation. Thus, the payoff of the traders would be as follows: $V(S_1) = p_1$, $V(B_1) = \frac{1}{2}(1-p_1) + \frac{1}{2}\left(\frac{\delta}{2}\right)$, $V(S_2) = \delta\frac{1}{2}\left(\frac{1+\lambda}{2}\right)$, $V(B_2) = \frac{1}{2}(\lambda - p_1) + \frac{1}{2}\left(\delta\frac{\lambda}{2}\right)$.

Notice that, if this allocation was a subgame perfect equilibrium, it would be the case that the following conditions were satisfied.

First, as usual it must be the case that $p_2 \geq p_1$, for otherwise both buyers had accepted the lower price p_2 instead.

Second, it must hold that the expected payoff for B_1 from accepting the same price which also B_2 is accepting is higher than the expected surplus attached to any of the possible alternatives.

In particular, if this was an equilibrium the expected payoff by accepting p_1 given that also B_2 accepts p_1, must be at least equal to the one in case B_1 accepted p_2 instead, given that B_2 accepts p_1, that is $\frac{1}{2}\left(1+\frac{\delta}{2}\right) - \frac{1}{2}p_1 \geq 1 - p_2$.

Analogously, the expected surplus by accepting p_1 given that also B_2 accepts p_1, must be at least equal to the one in case B_1 rejected both offer, given that B_2 accepts p_1. In the latter case, B_1 would be matched in the next period with the remaining seller, starting a bilateral negotiation: thus, if this was an equilibrium, it would be true that is $\frac{1}{2}\left(1+\frac{\delta}{2}\right) - \frac{1}{2}p_1 \geq \frac{\delta}{2}$.

Manipulating these inequalities gives the set of conditions for B_1's strategy being an equilibrium,

$$\begin{cases} p_1 \leq 1 - \frac{\delta}{2} \\ p_2 \geq \frac{1}{2}p_1 + \frac{1}{2}\left(1 - \frac{\delta}{2}\right) \end{cases},$$

which, together, also prove formally the first described condition, as $p_1 \leq \frac{1}{2}p_1 + \frac{1}{2}\left(1-\frac{\delta}{2}\right) \leq p_2$.

The same logic leads to the set of analogous conditions for B_2's strategy being an equilibrium:

$$\begin{cases} p_1 \leq \lambda\left(1 - \frac{\delta}{2}\right) \\ p_2 \geq \frac{1}{2}p_1 + \frac{1}{2}\lambda\left(1 - \frac{\delta}{2}\right) \end{cases}.$$

Now, notice that, as $\lambda < 1$, the first condition for B_2 implies clearly the strict inequality of the analogous condition for B_1, that is $p_1 < 1 - \frac{\delta}{2}$.

This in turn allows to further restrict the first condition on the relative size of the two prices. In fact, note that, if this allocation was an equilibrium, it must hold the strict inequality $p_2 > p_1$. To see why, assume at the contrary that $p_2 = p_1$. If this was the case it is immediate to show that B_1, taking as given the acceptance by B_2, would deviate by accepting p_2 instead. In fact, by deviating and accepting p_2 buyer B_1 would obtain $1 - p_2$ rather than $\frac{1}{2}\left(1 + \frac{\delta}{2}\right) - \frac{1}{2}p_1$. If it was true that $p_2 = p_1$, B_1 would deviate if and only if $1 - p_1 \geq \frac{1}{2}\left(1 + \frac{\delta}{2}\right) - \frac{1}{2}p_1$, which is always verified as we have found that in equilibrium must hold that $p_1 < 1 - \frac{\delta}{2}$. Thus if the above allocation was indeed an equilibrium it must also hold that $p_2 > p_1$.

However, the conditions that hold when the above allocation emerges, also contradict the assumption that the latter constituted a pure-strategies subgame perfect equilibrium.

In fact, consider the seller S_1. In the original allocation he offered $p_1 \leq \lambda\left(1 - \frac{\delta}{2}\right) < 1 - \frac{\delta}{2}$, that is a price such that the low-valuation buyer is indifferent between accepting it or rejecting it, while the high-valuation buyer is strictly better off accepting it. However, as also $p_1 < p_2$, seller S_1 may profitably deviate by proposing a price $p_1' = p_1 + \varepsilon$ which is still below p_2. In fact, in such a way, she makes a proposal that will be accepted by the high-valuation buyer only, giving her an higher payoff than the initial situation. Hence, also these allocations can never constitute a subgame perfect equilibrium, because of the incentive by the sellers to serve only the high-valuation buyer.

2.1.4 The Last Class: Both Buyers Accept Prices from Different Sellers.

Having eliminated all the other seven possible cases, the set of the potential candidate to subgame perfect equilibria has drastically restricted to just two remaining cases, which are situations where both buyers accept prices from different sellers,

$$\begin{array}{cc} S_1 & S_2 \\ p_1 & p_2 \\ B_1 & B_2 \end{array} \quad ; \quad \begin{array}{cc} S_1 & S_2 \\ p_1 & p_2 \\ B_2 & B_1 \end{array} \; :$$

again we refers to the first case in the figure, being the treatment for the other completely symmetric.

In such a case, all the goods would be immediately sold and each buyers would trade with a different seller, possibly at different prices. The payoffs of the traders would be as follows: $V(S_1) = p_1$, $V(B_1) = 1 - p_1$, $V(S_2) = p_2$, $V(B_2) = \lambda - p_2$.

Notice that, if this allocation was a subgame perfect equilibrium, it would be the case that the following conditions were satisfied.

First, each buyer should expect an higher payoff by accepting his price rather than rejecting it: in the latter case, given that the second buyer is accepting the offer from the other seller, the rejecting buyer would be matched in the next period with the same seller, starting a bilateral negotiation. Thus, clearly it must hold that $1 - p_1 \geq \frac{\delta}{2}$ and that $\lambda - p_2 \geq \delta\frac{\lambda}{2}$. These imply the following conditions on the prices $p_1 \leq 1 - \frac{\delta}{2}$ and $p_2 \leq \lambda\left(1 - \frac{\delta}{2}\right)$, which in turn imply that $p_2 < 1 - \frac{\delta}{2}$.

Second, if this was an equilibrium, it must be true that each buyer would expect an higher payoff by accepting his price rather than the price that the other buyer also accepts, in the latter case being randomly selected to buy the good from that seller only with probability $\frac{1}{2}$, while with the same probability going to bilateral negotiation with the former seller. That is, also the following conditions must hold: $1 - p_1 \geq \frac{1}{2}(1 - p_2) + \frac{1}{2}\left(\frac{\delta}{2}\right)$ and $\lambda - p_2 \geq \frac{1}{2}(\lambda - p_1) + \frac{1}{2}\left(\delta\frac{\lambda}{2}\right)$. These imply the further restrictions on the prices $p_1 \leq \frac{1}{2}\left(1 - \frac{\delta}{2} + p_2\right)$ and $p_2 \leq \frac{1}{2}\left[\lambda\left(1 - \frac{\delta}{2}\right) + p_1\right]$, which in turn, together with $p_2 < 1 - \frac{\delta}{2}$, imply that also $p_1 < 1 - \frac{\delta}{2}$.

This first set of conditions importantly restrict the characteristics of the potential equilibrium, as they rule out the possibility that both sellers symmetrically propose the price that makes the high-valuation buyer as good as in a bilateral negotiation. In other words, if this was the equilibrium, the high-valuation buyer always would gain an higher surplus than in a bilateral negotiation with one single seller, thus benefiting from the presence of the low-valuation buyer in the thin market.

This is not surprising, however, since the case in which both sellers set up prices so high to cut off the low-valuation buyer and to extract the same surplus from the high-valuation buyer as in a bilateral bargaining, can never be an equilibrium because of the competition among the sellers, and of their incentives to undercut, as illustrated in the Second Class.

We now show that neither this last Class may constitute a pure-strategies subgame perfect equilibrium.

Notice that, given the two conditions $p_1 < 1 - \frac{\delta}{2}$ and $p_2 \leq \lambda\left(1 - \frac{\delta}{2}\right)$ we must look for a candidate equilibrium only in the space strictly below the line $p_1 = 1 - \frac{\delta}{2}$ corresponding to the price that makes the high-valuation buyer indifferent between buying in the thin market or going to bilateral negotiation. Furthermore, we also must look for a candidate equilibrium where the price charged to the low-valuation buyer is never above line $p_2 = \lambda\left(1 - \frac{\delta}{2}\right)$, since, to be the equilibrium in this Class, must also be that the price charged to B_2 makes him not worse off than in bilateral negotiation.

Hence, we must rule out from the set of equilibrium candidates all the possible situations where *both* prices are above line $p_2 = \lambda\left(1 - \frac{\delta}{2}\right)$, as these correspond to the cases described in the Second Class where both sellers choose to serve only the high-valuation buyer.

Furthermore, it easy to show that if p_1 and p_2 were equilibrium price offers accepted respectively by B_1 and B_2, it must necessarily be that they would be equal: $p_1 = p_2$.

In fact, suppose at the contrary that in equilibrium the two sellers propose different prices to the buyers, that is $p_1 \neq p_2$. The idea is that the sellers might want to try to extract an higher price from the highest-valuation buyer. Thus, since it does not make much sense that the price charged to the highest-valuation buyer would be lower than the one charged to the low-valuation buyer, assuming $p_1 \neq p_2$ has to imply that we assume $p_1 > p_2$.

Notice that, as the condition $p_2 \leq \lambda\left(1 - \frac{\delta}{2}\right)$ must hold if this was an equilibrium, the seller S_2 is charging a price that is still accepted by the low-valuation buyer. However, given that S_1 is proposing a price $p_1 > p_2$, we now show that seller S_2 can indeed profitably deviate from her original strategy. In fact, three subcases are in order.

In the first special subcase, S_2 is setting a price $p_2 < \frac{\delta}{4}(1 + \lambda)$, while S_1 proposes any price $p_1 > p_2$. Note that in such a case the price set by seller S_2 is strictly lower than the average gain by going to bilateral negotiation with one from the two buyers. This implies that in such a case, seller S_2 have a profitable deviation by proposing any price $p_2' = p_1 + \varepsilon > p_2$, with $\varepsilon > 0$. In fact, in such a way, she may attempt to capture the demand of the high-valuation buyer knowing that the worse that can happen to her is that both buyers will accept p_1. But since in the latter case one of the two buyers is not matched with S_1 and is instead selected to go to bilateral negotiation with S_2, she indeed gets an higher payoff by deviating. This subcase, however, is not generally exhaustive, as depends on the specific values of δ and λ. In particular the value $\frac{\delta}{4}(1 + \lambda)$ may be either below or above the line $\lambda\left(1 - \frac{\delta}{2}\right)$ depending on the special configuration of parameters $\lambda \lesseqgtr \frac{\delta}{4-3\delta}$. The next two subcases, at the contrary, cover with generality all the set of the possibilities that may emerge.

In the second subcase, indeed, S_2 sets a price $p_2 < \lambda\left(1 - \frac{\delta}{2}\right)$, while S_1 proposes a price $\lambda\left(1 - \frac{\delta}{2}\right) \geq p_1 > p_2$. It is immediate to observe that this subcase can never be an equilibrium, because S_2 can indeed profitably deviate by proposing a price $p_2' = p_1 - \varepsilon > p_2$, with $\varepsilon > 0$ infinitesimally small, which will still be accepted at least by the low-valuation buyer.

In the third subcase, finally, S_2 sets a price $p_2 \leq \lambda\left(1 - \frac{\delta}{2}\right)$, while

S_1 proposes a price $p_1 > \lambda\left(1-\frac{\delta}{2}\right) \geq p_2$. It is easy to see that neither this subcase can ever be an equilibrium, as S_2 again may increase the price offered, by proposing a price $p_2' = p_1 - \varepsilon > p_2$, with $\varepsilon > 0$. As long as $p_2' > \lambda\left(1-\frac{\delta}{2}\right)$, this deviation is profitable for S_2: in fact, on the one hand, B_2 would no longer accept the price p_2', on the other hand, however, B_1 would prefer to buy from S_2 rather than from S_1 at a price $p_1 > p_2'$, and this clearly ensures S_2 an higher payoff than the original strategy.

Hence, if it existed an equilibrium in this Class, it would imply that two different prices would never be charged by the sellers.

The latter observation definitely rules out from the set of equilibrium candidates all the situations in which two different prices coexist in the thin market when the sellers are the proposers. In particular, it rules out the only one possibility left, given the conditions holding in an equilibrium, that is when S_2 is setting a price $p_2 = \lambda\left(1-\frac{\delta}{2}\right)$ while S_1 is charging on the high-valuation buyer an higher price $\lambda\left(1-\frac{\delta}{2}\right) < p_1 < 1-\frac{\delta}{2}$. Thus, if it existed an equilibrium in this Class, it would necessarily imply that the two sellers would be charging identical prices.

Also note that the case in which both sellers charge an identical price $\lambda\left(1-\frac{\delta}{2}\right) < p < 1-\frac{\delta}{2}$ is not compatible with the conditions describing an equilibrium in this class, and may never constitute an equilibrium because each seller, in order to capture the demand from the high-valuation buyer, has an incentive to deviate by proposing a price strictly below the one charged by the competitor.

Hence, only two possible cases remain to be analyzed as candidate to the equilibrium, that is either both sellers charge some identical price $p < \lambda\left(1-\frac{\delta}{2}\right)$ or both set a price $p = \lambda\left(1-\frac{\delta}{2}\right)$.

Consider first the latter case. Chatterjee and Dutta have claimed that is *"trivial to check"* that the latter case is the unique subgame perfect equilibrium of the S-game. However, we now show that in our model this is no longer true, as even in this case one of the seller has always an incentive to deviate.

In fact, suppose that in equilibrium both sellers set an identical price $p = \lambda\left(1-\frac{\delta}{2}\right)$. In such a case the two buyers are perfectly indifferent between which seller buying from. If the the described one was indeed an equilibrium, however, each buyer would have selected a different seller, for otherwise a profitable deviation would exist. The same result of *coordination to not coordinate* the purchases may be ensured by assuming the existence of an exogenous predetermined mechanism which randomly assigns a buyer to a seller in case of a tie in the price offers. In the following, then we assume, without any lack of generality, that the allocation is the one illustrated in the first case of the above figure, where buyer B_1

is accepting the offer from S_1 and buyer B_2 is accepting the offer from S_2.

One may well argue with Chatterjee and Dutta that the present is indeed a subgame perfect equilibrium, since the sellers has no profitable deviations. In fact, it is argued that if S_2 offered any price $p_2' < p$ she would support a lower payoff than in the initial situation. Alternatively if she proposed any price $p_2' = p + \varepsilon$, with $\varepsilon > 0$, buyer B_2, would be indifferent between rejecting both prices and accepting p from S_1. In the former case, S_2 would expect from the subsequent bilateral negotiation a payoff equal to $\delta \frac{\lambda}{2}$, which is never greater than the original $p_2 = \lambda \left(1 - \frac{\delta}{2}\right)$, while in the latter case, she would expect $\frac{\delta}{4}(1+\lambda)$ which is higher than $\lambda \left(1 - \frac{\delta}{2}\right)$ only for the special configuration of parameters $\lambda > \frac{\delta}{4-3\delta}$. Hence one may conclude that, for general values of the parameters δ and λ, there are no profitable deviations for seller S_2, which in turn proves that the present is indeed an equilibrium.

However, we now show that an allocation where both sellers set an identical price $p = \lambda \left(1 - \frac{\delta}{2}\right)$ can never be an equilibrium either. In particular, seller S_1, may always profitably deviate by proposing a price $p_1' = p + \varepsilon$, with ε sufficiently small. In fact, assume that she proposes such a price. The logic reported above, about an analogous deviation by S_2, would suggest that no buyer would ever accept such an higher price: either B_1 would reject both offers, or he would accept the lower $p = \lambda \left(1 - \frac{\delta}{2}\right)$ from S_2 instead.

However, first note that, responding to such a deviation by S_1, buyer B_1 would never choose to reject both offers: in fact, in the latter case, given that B_2 must accept p from S_2, he would be matched in a bilateral negotiation with S_1, from which he may expect a surplus $\frac{\delta}{2}$, that in turn is always strictly lower than the one he would get by accepting p from S_2, as $\frac{1}{2}\left[1 - \lambda\left(1 - \frac{\delta}{2}\right)\right] + \frac{1}{2}\left(\frac{\delta}{2}\right) > \frac{\delta}{2}$. Hence, when responding to such new price $p_1' = p + \varepsilon$ by S_1, buyer B_1 would never choose to reject both offers.

Then note that, for ε sufficiently small, buyer B_1 would always prefer to accept the price $p_1' = p + \varepsilon$ from S_1 rather than the price p from S_2, which is also accepted by the low-valuation buyer. The reason is that, in the latter case, he would expect $\frac{1}{2}\left[1 - \lambda\left(1 - \frac{\delta}{2}\right)\right] + \frac{1}{2}\left(\frac{\delta}{2}\right)$ from the subsequent random selection, which is strictly lower than $1 - \lambda \left(1 - \frac{\delta}{2}\right) - \varepsilon$ if ε is small enough.

Thus, we have shown that, by proposing a price $p_1' = p + \varepsilon$, seller S_1 may always profitably deviate from an original situation where both sellers propose an identical price $p = \lambda \left(1 - \frac{\delta}{2}\right)$: the latter then can not be an equilibrium. The reason is that, when both sellers charge the same price, there is always an incentive to deviate, in the attempt to capture the demand from the high-valuation buyer, by charging a marginally higher price that will be still accepted by the latter because of the competition exerted by the low-valuation buyer.

The same logic applies a fortiori to show that neither the situation where both sellers charge some identical price $p < \lambda\left(1-\frac{\delta}{2}\right)$ may constitute an equilibrium.

Hence, we have just shown that allocations where the two buyers accept from different sellers either identical or different prices, can not constitute a subgame perfect equilibrium.

This case definitely proves that in all the subgames of the original game starting with the selection of the sellers as proposers, that we denote S-games, the bargaining game among traders in a bilateral duopoly can not exhibit pure-strategies subgame perfect equilibria. To find such equilibria is then necessary to address attention to the exploration of mixed strategies by the sellers.

2.2 B-games

Consider now all the subgames of the original game starting with the selection of the buyers as proposers. In these subgames we denote p_1 and p_2 the price offered simultaneously and independently by the buyers B_1 and B_2 respectively.

We now describe the conditions for all the possible outcomes of the game to be subgame perfect equilibria. Analogously to the case of S-games, exactly 9 possible equilibrium allocations of the goods may emerge from a bargaining period in which the buyers make proposals.

We classify again the 9 possible allocations in 4 classes and we show that some of them can never represent a subgame perfect equilibrium of the S-games. We thus gradually restrict the set of the potential equilibria to fewer classes of cases. Finally, by having eliminated all the cases from the set, we show that in the B-games there exists a pure-strategies equilibrium where some trade happens with delay.

2.2.1 First Class: Both Sellers Reject Both Prices

The first case emerges where both sellers reject both the offers p_1 and p_2 by the two buyers. We may represent the candidate equilibrium by the figure

$$\begin{array}{cc} B_1 & B_2 \\ p_1 & p_2 \\ \varnothing & \varnothing \end{array}$$

where the last row indicates the set of the sellers accepting the price p_i by the buyer B_i, $i = 1, 2$.

In such a case all the players do not trade and enter the next round, with a new selection of the proposers. Their relative surplus are given by the discounted value of the expected payoff by entering a new stage of negotiation.

It is immediate to observe that this case can never constitute a subgame perfect equilibrium. In fact, if in the following round of bargaining the sellers will be proposed to make offers, no trade will be feasible as we have just shown that no subgame perfect equilibrium will never be reached. Hence, buyers will only be able to obtain some surplus from the trade only if they will be again selected to make offers in the following round, which happens with probability $\frac{1}{2}$. As we focus on stationary strategies by the traders, the set of the potential equilibrium payoffs for the traders remains the same in any B-game. Thus, if there existed any positive payoff the buyers may obtain by proposing prices, it would be clearly better for the buyers to propose sooner than later, as delays are costly. But it is also clear that such a positive payoff do exist, as each buyer may always propose at least the price emerging in bilateral negotiations, which will be accepted by the sellers. Thus, this case can never constitute an equilibrium.

2.2.2 Second Class: One Seller Accepts a Price, the Other Rejects Both Offers

The second possible situation emerges when only one from the two sellers accepts one price, while the other rejects both. This situation includes four cases, depending on the identities of the seller who accepts and of the buyer who proposes the price:

$$\begin{array}{cc} B_1 & B_2 \\ p_1 & p_2 \\ S_1 & \varnothing \end{array} \;;\; \begin{array}{cc} B_1 & B_2 \\ p_1 & p_2 \\ \varnothing & S_1 \end{array} \;;\; \begin{array}{cc} B_1 & B_2 \\ p_1 & p_2 \\ S_2 & \varnothing \end{array} \;;\; \begin{array}{cc} B_1 & B_2 \\ p_1 & p_2 \\ \varnothing & S_2 \end{array} .$$

Given the symmetry of the game, we only consider the allocations as represented by the first and the second figures, then adapting the findings to the other identical seller.

In such cases, only one seller trade immediately with a buyer at the proposed price, while the other seller enters, in the following period, a new bilateral negotiation with the remaining buyer. We model the latter negotiation as a Rubinstein bilateral bargaining with random selection of the proposer at every period. Hence, both the remaining seller and the unmatched buyer expect from the bilateral negotiation one-half of the possible surplus to be divided. Thus, in the first case both B_2 and S_2 each expects a discounted payoff of $\delta\frac{\lambda}{2}$, while in the second case, both B_1 and S_2 each expects a discounted payoff of $\delta\frac{1}{2}$.

First note that the second and the fourth cases represented in the figure, in which the only accepted price is the one proposed by the low-valuation, intuitively can never be an equilibrium. In fact, it must be always the case that, if the buyer with the lowest valuation can propose a price that will be accepted, this might be a fortiori proposed also by the highest valuation buyer.

The last two cases represented in the figure indeed do not make much sense, since it is clear that the high-valuation buyer can always deviate by proposing a price $p'_1 = p_2 + \varepsilon$, then attracting the seller that is already accepting p_2. Thus these cases can never be equilibria.

Furthermore, note that the perfect symmetry among the sellers raises another major question about allocations within this class: how can it be possible that two identical sellers behave differently, one accepting and the other rejecting the same offer in equilibrium?

We now show that no case in this class can indeed be a subgame perfect equilibrium. The reported proof refers to the first case in the figure, but it clearly extends by symmetry to the third case, and, a fortiori, to the other two.

Consider the case where, as the outcome of the negotiation, buyer S_1 accepts the price proposed by seller B_1, while buyer B_2 rejects both the prices offered by the two sellers.

The resulting allocation of the goods would be that S_1 sells to B_1 at price p_1, while the low-valuation seller would trade in a bilateral negotiation with B_2 after some delay. The resulting expected payoffs from such an allocation would be $V(S_1) = p_1$, $V(B_1) = 1 - p_1$, $V(S_2) = V(B_2) = \delta \frac{\lambda}{2}$.

Notice that, if this allocation was a subgame perfect equilibrium, it would be the case that the following conditions were satisfied.

First, it must be the case that $p_2 < p_1$. In fact if it was that $p_2 > p_1$, seller S_1 would have accepted the higher price p_2 instead. Again, if the buyers were proposing the same price, so that $p_2 = p_1$, then, given that S_1 accepts p_1, in an equilibrium the symmetry by the sellers would ensure that S_2 will accept p_2. Note that, as $p_2 < p_1$, from the same argument of the sellers' symmetry we should expect also S_2 will accept p_2, which in fact intuitively contradicts the present allocation being an equilibrium.

Second, for the price p_1 to be accepted by seller S_1 it must be set to a level such that the latter is indifferent between accepting it, gaining p_1, and rejecting it going to a further negotiation round in a situation such as the one described in the First Class.

Third, must be the case that, by rejecting both offers, buyer S_2 expected an higher payoff than by accepting one of the two. In particular, if seller S_2 accepted the same offer p_1, he would be randomly selected with probability $\frac{1}{2}$ to sell the good rather than going to bilateral negotiations with the remaining buyer. Then if this was an equilibrium it must be that the expected payoff for S_2 by accepting p_1 would never be as high as the payoff he may obtain by rejecting and going directly to bilateral negotiation with B_2, that is the following must hold: $\frac{1}{2}p_1 + \frac{1}{2}\left(\delta\frac{\lambda}{2}\right) \leq \delta\frac{\lambda}{2}$.

The latter implies that the emerging price would be such that $p_1 \leq \delta\frac{\lambda}{2}$, which in turn also implies, with the first condition, that $p_2 < \delta\frac{\lambda}{2}$, a condition that guarantees that buyer S_2 never accepted price p_2.

Having derived these conditions to hold when the above allocation emerges, one may verify that they contradict the assumption that the latter can ever constitute a pure-strategies subgame perfect equilibrium.

In fact, consider buyer B_2. If the present allocation was the equilibrium he would earn a surplus $\delta\frac{\lambda}{2}$. Consider now a deviation by B_2 from the described strategy. For instance, he may deviate by proposing a price $p_2' = p_1 + \varepsilon$, with ε infinitesimally small and, clearly, $p_1 \leq \delta\frac{\lambda}{2}$. We now show that this is indeed a profitable deviation, as B_2 will surely convince S_1 to sell him the good at that price, being able to earn $\lambda - p_1 - \varepsilon$ rather than the lower $\delta\frac{\lambda}{2}$. In fact, the condition $p_1 \leq \delta\frac{\lambda}{2}$ implies that $p_2' = p_1 + \varepsilon \leq \delta\frac{\lambda}{2} + \varepsilon$ and then that $\lambda - p_1 - \varepsilon \leq \lambda - \delta\frac{\lambda}{2} - \varepsilon$. However, as ε is so small that $\varepsilon \leq \lambda(1-\delta)$, that is $\varepsilon \longrightarrow 0$ as $\delta \longrightarrow 1$, it always holds that $\lambda - p_1 - \varepsilon \geq \delta\frac{\lambda}{2}$, which in turn implies that, by proposing p_2', B_2 may convince seller S_1 to sell him the good and can profitably deviate from the original situation.

Thus, the described allocations can never constitute a subgame perfect equilibrium, because of two forces: the symmetry among the sellers makes impossible for the buyers to offer a price that would be accepted by only one of them, and also both buyers have incentives to propose offers that guarantee themselves a payoff no lower than in bilateral negotiations. For the other three cases in the figure, the same logic applies, with the important consideration that, such as in an auction, the high-valuation buyer may always offer an higher price than the lowest-valuation buyer.

2.2.3 Third Class: Both Sellers Accept Prices from Different Buyers.

The third class embraces two symmetric situations where both sellers accept prices from different buyers,

$$\begin{array}{cc} B_1 & B_2 \\ p_1 & p_2 \\ S_1 & S_2 \end{array} \quad ; \quad \begin{array}{cc} B_1 & B_2 \\ p_1 & p_2 \\ S_2 & S_1 \end{array} \; :$$

again we refers to the first case in the figure, being the treatment for the other completely symmetric.

In such a case, all the goods would be immediately sold and each buyer would trade with a different seller, possibly at different prices. The payoffs of the traders would be as follows: $V(S_1) = p_1$, $V(B_1) = 1 - p_1$, $V(S_2) = p_2$, $V(B_2) = \lambda - p_2$.

Notice that, if this allocation was a subgame perfect equilibrium, it would be the case that the following conditions were satisfied.

First, each seller should expect an higher payoff by accepting his price rather than rejecting it: in the latter case, given that the second seller is accepting the offer from the other buyer, the rejecting seller would be matched in the next period with the same buyer, starting a bilateral negotiation. Thus, clearly it must hold that $p_1 \geq \frac{\delta}{2}$ and that $p_2 \geq \delta\frac{\lambda}{2}$. These imply that $p_1 > \delta\frac{\lambda}{2}$.

Second, if this was an equilibrium, it must be true that each seller would expect an higher payoff by accepting her price rather than the price that the other seller also accepts, in the latter case being randomly selected to sell the good to that buyer only with probability $\frac{1}{2}$, while with the same probability going to bilateral negotiation with the former buyer. That is, also the following conditions must hold: $p_1 \geq \frac{1}{2}p_2 + \frac{1}{2}\left(\frac{\delta}{2}\right)$ and $p_2 \geq \frac{1}{2}p_1 + \frac{1}{2}\left(\delta\frac{\lambda}{2}\right)$. By using the fact that $\lambda < 1$, these two inequalities imply two further conditions to hold if this was an equilibrium:

$$\begin{cases} p_2 - \frac{1}{2}p_1 \geq \delta\frac{\lambda}{4} \\ \frac{1}{2}p_1 > \delta\frac{\lambda}{4} \end{cases}$$

which in turn imply, after some manipulations, that $p_1 > \delta\frac{\lambda}{2}$ and that $p_2 > \delta\frac{\lambda}{2}$.

This set of conditions show that, if this was an equilibrium, it would imply that the price offered by B_2 would be higher than the payoff for S_2 by going to bilateral negotiation with him. That is, if this was an equilibrium, it would imply that the low-valuation buyer would allow the seller which he trades with to gain an extra-profit with respect to her outside option. But then it may immediately observed that this strategy can not constitute an equilibrium for B_2. In fact the low-valuation buyer, taking as given that the price offered by B_1 is $p_1 > \delta\frac{\lambda}{2}$, may profitably deviate by marginally decreasing his offer, that is by proposing a price $p_2' = p_2 - \varepsilon \geq \delta\frac{\lambda}{2}$ which will be still accepted by the seller S_2. This contradicts the assumption that the present allocation is an equilibrium.

2.2.4 Last Class: Both Buyers Accept a Price from the Same Seller

Having eliminated all the other seven possible cases, the set of the potential candidate to subgame perfect equilibria has drastically restricted to just two remaining cases, which are situations where both buyers accepts the price offer from the same seller. This situation includes two asymmetric cases, that must be treated separately:

$$\begin{array}{ll} B_1 & B_2 \\ p_1 & p_2 \\ S_1, S_2 & \varnothing \end{array} \quad ; \quad \begin{array}{ll} B_1 & B_2 \\ p_1 & p_2 \\ \varnothing & S_1, S_2 \end{array} .$$

In the first case, both sellers accept the offer by the high-valuation buyer. In such a case, one of the sellers would be randomly selected to trade with B_1, while the other will be matched in the next period with the remaining buyer, starting a bilateral negotiation. Thus, the payoff of the traders would be as follows: $V(S_1) = V(S_2) = \frac{1}{2}p_1 + \frac{1}{2}\left(\delta\frac{\lambda}{2}\right)$, $V(B_1) = 1 - p_1$, $V(B_2) = \delta\frac{\lambda}{2}$.

In the second case, both sellers accept instead the offer by the low-valuation buyer. Again, one of the sellers would be randomly selected to trade with B_2, while the other will be matched in the next period with the high-valuation buyer, starting a bilateral negotiation. Thus, the payoff of the traders would be as follows: $V(S_1) = V(S_2) = \frac{1}{2}p_1 + \frac{1}{2}\left(\frac{\delta}{2}\right)$, $V(B_1) = \frac{\delta}{2}$, $V(B_2) = \lambda - p_2$.

We first treat this second case, immediately observing that it can never be an equilibrium. The intuition is clear: any price that may be offered by the low-valuation buyer may be afforded by the high-valuation buyer as well, so that the latter may always undermine the present allocation by attracting at least one seller, thus gaining a surplus certainly greater than in a bilateral negotiation.

A formal proof is immediate. In fact, if the present was an equilibrium, the condition $\frac{1}{2}p_2 + \frac{1}{2}\left(\frac{\delta}{2}\right) \geq \frac{\delta}{2}$ would necessarily hold since each seller would be better off by accepting the price p_2 , given that also the other seller is accepting it, rather than rejecting both the offers, then going to bilateral negotiation with the high-valuation buyer. The latter condition would imply $p_2 \geq \frac{\delta}{2} > \delta\frac{\lambda}{2}$.

Furthermore, note that if this was an equilibrium, it would also necessarily be that $p_1 < p_2$. In fact, if, at the contrary, it was that $p_1 > p_2$, then both sellers would have accepted p_1 instead. If, again, it was that $p_1 = p_2$, one of the two seller, given the choice of the other, would have preferred to deviate by accepting p_1, rather than p_2: in fact, in this case, the condition $p_2 \geq \frac{\delta}{2}$ immediately implies that $p_1 \geq \frac{1}{2}p_2 + \frac{1}{2}\left(\frac{\delta}{2}\right)$.

Again, notice that if the present was an equilibrium, buyer B_2 would have chosen a price $p_2 = \frac{\delta}{2}$, as any higher price, while still accepted by both sellers, would imply a lower surplus for himself.

Hence, it may be observed that the present allocation can never be an equilibrium. In fact, the high-valuation buyer may always profitably deviate by proposing a price $p_1' = p_2 + \varepsilon$, with $\varepsilon > 0$: in fact, by doing so he may obtain a payoff of $1 - \frac{\delta}{2} - \varepsilon$ which is greater than the one he would get in bilateral negotiation as $1-\varepsilon > \delta$ for ε small enough.

Thus, if any subgame perfect equilibrium there exists in the B-games, it must be an allocation such as the only one left, where both sellers accept the same price from the high-valuation buyer. We now in fact show that a pure-strategies subgame perfect equilibrium, belonging to the allocation described by the first figure, does exists when the impatience rate δ assumes values sufficiently high. The equilibrium is as follows.

A complete and formal description of the already familiar sellers' response strategies to offer p_1 from B_1 and p_2 from B_2 is the following:

- If $p_1 < \delta\frac{\lambda}{2}$, reject any offer p_1.
- If $p_1 \geq \delta\frac{\lambda}{2} > p_2$, accept any offer p_1.
- If $p_1 < \delta\frac{\lambda}{2} \leq p_2$ and $p_2 \geq \frac{\delta}{2}$, accept p_2.
- If $p_1 < \delta\frac{\lambda}{2} \leq p_2$ and $p_2 < \frac{\delta}{2}$, accept p_2 if the other buyer rejects both offers, and reject both offers if the other buyer accepts p_2.
- If $p_1 \geq \delta\frac{\lambda}{2}$, $p_2 \geq \delta\frac{\lambda}{2}$ and $\frac{1}{2}\left(p_1 + \delta\frac{\lambda}{2}\right) \geq p_2$, accept p_1.
- If $p_1 \geq \delta\frac{\lambda}{2}$, $p_2 \geq \delta\frac{\lambda}{2}$, $\frac{1}{2}\left(p_1 + \delta\frac{\lambda}{2}\right) < p_2$ and $\frac{1}{2}\left(p_2 + \frac{\delta}{2}\right) \geq p_1$, accept p_2.
- If $p_1 \geq \delta\frac{\lambda}{2}$, $p_2 \geq \delta\frac{\lambda}{2}$ and $p_1 \geq \frac{\delta}{2}$, accept p_1.

Denote p_1^* and p_2^* the prices offered by B_1 and B_2 respectively. Then, the following price offers describe the subgame perfect equilibrium strategies by the buyers:

$$\begin{cases} p_1^* \text{ s.t. } \frac{1}{2}\left(p_1^* + \delta\frac{\lambda}{2}\right) = \lambda\left(1 - \frac{\delta}{2}\right) \\ p_2^* = \lambda\left(1 - \frac{\delta}{2}\right) \end{cases}.$$

We now check that these strategies constitute a subgame perfect equilibrium in which B_1's offer is immediately accepted by both sellers.

First rewrite explicitly the equilibrium price offered by the high-valuation buyer as $p_1^* = \frac{\lambda}{2}(4-3\delta)$. Then it may easily be checked that in equilibrium the buyers always propose different prices $p_1^* > p_2^*$. Thus it is indeed optimal for each seller to accept the offer p_1^* from the high-valuation buyer.

As both sellers accept the same price p_1^*, one of them will be randomly selected to sell to B_1 at the price p_1^*, while the other will be matched in the next period with the low-valuation buyer. Hence, the expected payoff by both sellers equals $\frac{1}{2}p_1^* + \frac{1}{2}\left(\delta\frac{\lambda}{2}\right) = \lambda\left(1 - \frac{\delta}{2}\right)$. Correspondingly, the low-valuation buyer B_2 obtains from the subsequent bilateral negotiation a discounted payoff of $\delta\frac{\lambda}{2}$.

Consider now if it does exist any profitable deviation by any of the traders. The sellers are clearly responding with an optimal strategy, as, by accepting both p_1^* and going to random matching, they expect the same payoff than by opting for p_2^*.

Consider then the low-valuation buyer. If B_2 deviates by proposing an higher price $p_2' = p_2^* + \varepsilon$, the latter price ensures to the sellers a payoff $\lambda\left(1 - \frac{\delta}{2}\right) + \varepsilon$ higher than by both choosing p_1^*: hence, at least one seller would deviate by accepting the new offer p_2'. However, such a deviation is clearly not profitable for B_2 as he gets $\lambda - \lambda\left(1 - \frac{\delta}{2}\right) - \varepsilon = \delta\frac{\lambda}{2} - \varepsilon$ which is lower than his surplus in bilateral negotiation.

If, on the other hand, B_2 deviates by proposing a lower price $p_2' = p_2^* - \varepsilon$, both the sellers will reject the offer, as they expect a payoff of only $\lambda\left(1 - \frac{\delta}{2}\right) - \varepsilon$ by accepting it. Hence, B_2 does not have indeed any profitable deviation.

Consider finally the high-valuation buyer. If B_1 deviates and makes an higher price $p_1' = p_1^* + \varepsilon$, the offer will be immediately accepted by both sellers, but B_1 will end up paying an higher price, which is clearly not profitable.

On the other hand, what happens if B_1 deviates and offers a lower price $p_1' = p_1^* - \varepsilon$? Each seller will observe that, by still accepting p_1' and going through random matching, she now may gain only $\lambda\left(1 - \frac{\delta}{2}\right) - \varepsilon$, that is less than what she gets by opting for p_2^*. Thus, both sellers will choose offer p_2^* instead.

In fact, in such a case, one of the two seller will be randomly selected to sell to the low-valuation buyer at price p_2^*, while the other will be matched with the high-valuation buyer in the next period: the expected payoff for each seller is then $\frac{1}{2}\left[\lambda\left(1 - \frac{\delta}{2}\right) + \frac{\delta}{2}\right]$, which is always greater than $\lambda\left(1 - \frac{\delta}{2}\right) - \varepsilon$ and, for values of δ sufficiently high, is also greater than $\frac{\lambda}{2}(4 - 3\delta) - \varepsilon$. Hence it is indeed optimal for both sellers to choose offer p_2^* after such a deviation by B_1.

Then it remains to check whether for B_1 the payoff by proposing a price $p_1' = \frac{\lambda}{2}(4 - 3\delta) - \varepsilon$ and going to bilateral negotiation with one of the seller it may be better than proposing the original price p_1^*. That is, we want to show that the payoff gained by the high-valuation buyer by proposing the equilibrium price is at least as high as the surplus he would obtain in bilateral bargaining: $1 - \frac{\lambda}{2}(4 - 3\delta) \geq \frac{\delta}{2}$. It is immediate to see that, as $1 > \lambda > \frac{1}{2}$, the latter condition is indeed always verified. Hence, neither B_1 does have any profitable deviation. Thus we have shown that the described one is in fact a pure-strategies subgame perfect equilibrium for values of δ sufficiently high.

3 Conclusions

Thus we (should) have proved that the described bargaining game among two identical sellers and two heterogeneous buyers exhibits a unique pure-strategy subgame perfect equilibrium whenever the buyers are selected to make an offer (B-games) and no subgame perfect equilibrium in pure strategies when the sellers are selected to make an offer (S-games).

In particular, in the unique equilibrium in the B-games, both sellers accept immediately the price offered by the high-valuation buyer. Because of the random selection of the seller entitled to trade with the high-valuation buyer, one seller and the low-valuation buyer trade only with delays at a different price. Thus *both different prices and inefficiency due to costly delays emerge in the equilibrium.*

Moreover serious allocative inefficiency emerges in the thin market whenever the sellers are selected to make offers: endogenous negotiations drastically imply no trade at all, whereas transactions among the traders would clearly be Pareto-efficient.

These preliminary findings, if confirmed by further - and better - theoretical analysis and by some experimental evidence, seem to seriously undermine the pretence to strategically micro-found the (imperfectly) competitive equilibrium in thin markets on endogenous non-cooperative bargaining among the traders, along the line already proposed for large decentralized market. The existence of bilateral market power, as leads to more ambiguous results than the traditional analysis of oligopolistic markets, should deserve a further investigation.

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