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OPTIMAL INCOME TAXATION AND PUBLIC PROVISION OF PRODUCTIVE INPUTS

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Abstract

We characterize optimal non-linear income tax and optimal public provision of a productive input when gross earnings are unobservable and determined by exogenous productive capability and endogenous input investment. In the benchmark two-classes case, the public provision is welfare improving with respect to pure non-linear taxation only if the poor opt for the public program for lower levels of publicly provided input than the rich. Otherwise, pure non-linear taxation is the only relevant redistribution tool. The optimal public provision scheme is pure opting-out unless self-selection constraints require too low publicly provided input. The optimal public provision program does not affect the shape of optimal non-linear income taxation, unless self-selection constraints limit the scope for public provision. When self-selection constraints bind, the optimal taxation schedule can involve further marginal taxation of poor labor-supply (if public provision is too high) or marginal subsidization of rich labor-supply (if public provision is too low).

Keywords: In-kind redistribution; Non-linear income tax; Public provision of private goods; Opting out; Topping up

JEL classification: H42, H21

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1 Introduction

The second-best analysis of the public provision of private goods has challenged the traditional first-best view about redistribution, forging a new argument to justify public social services (Balestrino, 1999, 2000). Whenever household economic condition is imperfectly verified and affects the demand of some goods, in-kind transfers (or quotas on consumption) of these goods Pareto-dominate cash as a redistribution tool (Nichols and Zeckhauser, 1982; Guesnerie and Roberts, 1984). Efficiency gains are driven by higher costs of opportunistic behaviors in taking up subsidies (Blackorby and Donaldson, 1988) and in paying taxes (Guesnerie and Roberts, 1984), which in turn are determined by specific public provision rules.

The private goods that are usually provided by governments across the world have a relevant impact on households' earnings capacity: education and healthcare primarily affect human capital; also, services such as childcare and elderly-care can be considered inputs of households' income capacity. In this paper, we extend Boadway and Marchand (1995) and Greco (2011) models, to investigate the interaction between optimal provision schemes of productive social services and non-linear optimal taxation. Our benchmark model relies on a simple second-best framework with endogenous labor supply and nonlinear income taxation. Household income is the product of wage and labor supply. Following Boadway and Marchand (1995, p. 51-55), wage is modeled as an increasing function of two factors: exogenous production capability (or wealth) and productivity-enhancing input that can be provided by government and by private firms.

Inherited capability represents a composite asset summing up physical, financial, human and social "exogenous" factors, which determine household potential productivity; input represents any service or intermediate good that improves, in different ways, household potential earnings (e.g., education, healthcare, childcare or elderly-care). Depending on the assumed nature of the productivity-enhancing technology, input can be complementary to or substitute of exogenous capability. When household capacity to exploit potential earnings is strengthened by input (e.g., childcare or higher education) then it complements exogenous capability. Conversely, input substitutes capability when it affords households with more of the same factors constituting their production capacity (e.g., social network through schooling and basic living conditions through social housing or health prophylaxis).

Taking pure taxation as a benchmark, the public provision of input reduces the efficiency cost of redistribution in two ways. First, the public provision may alleviate tax distortion on household investment choices by forcing households to use more input (Boadway and Marchand, 1995; Cremer and Gahvari, 1997). Second, if transfers of input can be targeted to the poor more effectively than cash, then they improve the redistribution capacity of public policies by reinforcing self-selection mechanisms (Besley and Coate, 1991; Munro, 1992; Blomquist and Christiansen, 1995).

Public provision policies can be implemented by two schemes. First, government can support private expenditure on social services by a so-called *topping-up* scheme: all households receive conditional transfers (e.g., vouchers, tax allowances) to buy a given quality or quantity of the considered good; and households can top up public provision. Second, government can offer public services as an alternative to private services (so-called *opting-out* schemes): households are free to choose private or public services (without private supplementing).[1] Mixed schemes - where topping-up and opting-out mechanisms coexist - can be observed in real-world public social programs.[2] Thus, a general representation of public provision schemes is featured by two pillars: the first (topping-up) pillar affords all households with some input that can be privately supplemented; the second (opting-out) pillar provides some additional input as an alternative to private supplementing.

In the benchmark case, where only two classes of households are considered, we find

[1] Private supplementing can be technologically unfeasible (e.g., children can attend only one school at a time) or legally forbidden.

[2] For example, households opting for a public school automatically give up tax credits for private schools; hence, savings on tax credits implicitly finance public school expenditure. Also, an increase of tax allowances for households' expenditure on private schools implicitly reduces the additional transfer (with respect to tax allowances) underlying free public schools.

that public provision is welfare improving, only if low-capability (or poor) households choose the second-pillar public provision for lower levels of publicly provided input with respect to high-capability (or rich) households. If the rich enter the second-pillar program for lower levels of publicly provided inputs, the only effective redistributive tool is non-linear taxation. The structure of optimal provision schemes depends on two structural features of the economy: the economic complementarity or substitutability of input and capability (i.e., if input demand increases or decreases in exogenous capability) and the balance between households' production heterogeneity and government's preference for redistribution. When households' heterogeneity is strong enough (as compared to government's preference for redistribution), discriminating policies - in which rich benefit and poor receive different levels of publicly provided input - are optimal. The prevailing optimal scheme is pure opting out. The publicly-provided input is lower (or higher) than what low-capability households would have chosen on their own, if input and capability are economic complements (or substitutes). The intuition is that when input and capability are complements, the rich potentially mimicking the poor is hurt by any further reduction in input provision; conversely, when input and capability are substitutes the rich mimicking the poor is hurt by any further increase in publicly provided input. When input-capability economic substitutability is too strong, the optimal level of publicly provided input can be beyond the level that induces also the rich to opt in the second pillar. In this case, the optimal provision scheme is pure topping-up.

In the benchmark case, the shape of the optimal non-linear taxation schedule is unaffected by the public provision scheme unless the scope of the second pillar provision is limited by the aim to discriminate among classes. When the optimal second-pillar provision is too low (i.e., when input and capability are strong economic complements), then the marginal tax wedge on low-capability labor supply is wider than in the pure taxation case. The intuition is that the second-pillar provision is too much favorable to the rich potentially mimicking the poor, thus an additional marginal tax distortion is required to satisfy the incentive compatibility constraint. Conversely, when the optimal second-pillar provision is too high (i.e., when input and capability are strong economic substitutes), then the optimal tax schedule corrects the incentive balance of the rich by subsidizing at the margin high-capability labor supply.

We extend these results to the case of a continuum of households' capabilities [INCOMPLETE HERE]

The paper is organized as follows. The Section 2 considers the related literature. The Section 3 presents the model and the relevant policy regimes. The Section 4 analyzes the different policy regimes and discusses the optimal policies in the benchmark two-classes case. The Section 5 analyzes the optimal policies in the extended multi-classes case. Finally, the Section 6 draws conclusions.

2 Related Literature

Our results complement the main findings of the literature on public provision of private goods, providing a somewhat different perspective. We consider social programs characterized by different and potentially coexisting provision rules, generalizing the approach of Blomquist and Christiansen (1998a), who contrast pure topping-up and opting-out schemes. The main difference with the literature regards the effect of households' heterogeneity on the scope for public provision. Consistent to the literature, we find that the degree of heterogeneity affects the optimal structure of social programs. Nevertheless, we show that the public provision of input plays a tax-correction role for any specification of technology, particularly when input demand does not depend on household private capability. This is at odds with the results highlighting that there is no scope for the public provision of a given commodity whenever households' heterogeneity has no effect on its demand (Balestrino, 1999, 2000).

The reason for this theoretical divergence is that, in our model, government provides an input that directly corrects tax distortion on household production effort. When household production function is separable, input demand is independent of household capability, and its public provision becomes a perfect tool for correcting tax distortion: in this case, public provision without private supplementing implements the first-best outcome. Conversely, in models with public provision of consumption goods, the government provides such goods to influence the productive effort of individuals (i.e., labor supply) indirectly. Therefore, once this indirect link is broken (i.e., when, by separability of utility function, the demand of the considered commodity does not depend on leisure), there is no more scope for public provision (Blomquist and Christiansen, 1998a, p. 405).

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3 The Model

The economy is populated by a large number (a unit measure) of households. The productivity of household i depends on exogenous individual capability θ_i and on investment in input q_i: $w_i = w(\theta_i, q_i)$, where $w(.,.)$ is strictly increasing and twice differentiable in both arguments, and concave in q. In our benchmark case, we assume that $\lambda \in (0,1)$ households have low capability ($\theta_i = \underline{\theta}$), while the others have high capability ($\theta_i = \overline{\theta}$). Also, we assume constant returns to scale production technology, competitive labor market (hence, gross wage is equal to individual productivity), and unitary price of investment and consumption.

The government maximizes the sum of households utilities, but it can only observe gross income $y_i = w_i \cdot l_i$, conversely gross wage rate w_i and household labor supply l_i are not observed. Government can observe after-tax income x_i, but not the exact amount of private consumption c_i and private investment in input q_i^m.

The input is also publicly provided. The government supplies a uniform quantity q^f through a first pillar (topping up scheme) independently of households' consumption and investment choices. Then, a supplementary quantity of input q^s is provided to individuals opting for a second public pillar, and accepting not to privately top-up the public provision.

Individuals opting out of the second pillar can privately supplement the first pillar input provision (with q_i^m).

The government budget constraint is $\lambda \cdot (\underline{y} - \underline{x}) + (1-\lambda) \cdot (\overline{y} - \overline{x}) \geq q^f + q^s \cdot I$, where: $\underline{y}$, $\underline{x}$, $\overline{y}$, and $\overline{x}$ are gross and net incomes of high-capability and low-capability households, respectively; and $I \in [0,1]$ is the share of population covered by the second-pillar public provision of input.

The utility function of the generic household is $U(c_i, l_i)$, strictly increasing and concave in private consumption, and strictly decreasing and concave in labor supply. Moreover, consumption and leisure are normal goods. By $l_i = \frac{y_i}{w_i}$, $c_i = x_i - q_i^m$ (for households opting out of the second-pillar provision), and $c_i = x_i$ (for households opting for the second-pillar provision), the generic utility of opting-out households is $U(x_i - q_i^m, \frac{y_i}{w(\theta_i, q^f + q_i^m)})$ and the utility of opting-in households is $U(x_i, \frac{y_i}{w(\theta_i, q^f + q^s)})$.

The timing of the model reads as follows. The government determines the non-linear income tax and the first- and second-pillar public provision of input, then households choose gross- and net-of-tax incomes and whether to opt out of the second-pillar provision, and in this case the amount of private supplement of the first-pillar provision.

3.1 Household's choices and Policy regimes

In this section we consider the effect of government policies on the household's behavior. Taking as given gross and net incomes, we first analyze input private demand of opting-out households, then household decision to opt out or to accept the second-pillar provision. Finally, we discuss the behavior of households with different capabilities, reacting to different tax schedules.

3.1.1 Private demand of input

Let us assume that a generic household with capability θ decided to opt out. Taking as given gross and net incomes, let $V(x, y, q^f, \theta) \equiv \max_{q^m} U(x - q^m, \frac{y}{w(\theta, q^f + q^m)})$. If $q^m > 0$, by comparative statics of the first order condition with respect to q

$$-(U_c + U_l \cdot \frac{y}{w^2} \cdot w_q) = 0 \qquad (1)$$

it is possible to show that - under the assumed well-behaved preferences and technology - the private demand of input, $q^m(x, y, q^f, \theta)$, is increasing in net income ($\frac{dq^m}{dx} \in (0,1)$) and in gross income ($\frac{dq^m}{dy} \in (0, \frac{dx}{dy} |_V)$), and it is decreasing in first-pillar public provision - though there isn't complete crowding out: $\frac{dq^m}{dq^f} \in (-1, 0)$. The behavior of private input demand with respect to exogenous ability is such that:

1. if input and capability are strong technologic complements (say, there is some $w^+_{q\theta} > 0$ such that $w_{q\theta} > w^+_{q\theta}$), then private input is an economic complement of capability ($\frac{dq^m}{d\theta} > 0$);

2. if input and capability are strong technologic substitutes (say, there is some $w^-_{q\theta} < 0$, such that $w_{q\theta} < w^-_{q\theta}$), then private input is an economic strong substitute of capability ($\frac{dq^m}{d\theta} < -\frac{w_\theta}{w_q}$[3]);

3. if capability and input are weak technologic complements or substitutes (say, $w_{q\theta} \in (w^-_{q\theta}, w^+_{q\theta})$), then private input is an economic substitute of capability ($\frac{dq^m}{d\theta} \in (-\frac{w_\theta}{w_q}, 0)$).

It is worth remarking that to observe economic complementarity between capability and private demand for input, other things equal, we need strong technologic complementarity. In the other cases, the effect of capability on the private demand for input is negative: high-capability households would - other things equal - optimally demand less input. In the Appendix, we show that the Single Crossing Property (SCP)

$$\frac{d}{d\theta}(\frac{dx}{dy} |_V) = \frac{\partial}{\partial\theta}(\frac{dx}{dy} |_V) + \frac{\partial}{\partial q}(\frac{dx}{dy} |_V) \cdot \frac{dq^m}{d\theta} < 0$$

[3]Remark that $-\frac{w_\theta}{w_q}$ is the marginal rate of technologic substitution keeping constant the level of w.

taking into consideration the effect of θ on private input demand, is never satisfied in the case of strong substitutability between input and capability (case 2), therefore we will exclude such a case in our analysis.[4]

3.1.2 Opting out choice

We consider now the opting out choice.[5] The opting-in condition of the generic household can be written as

$$q^s(x, y, q^f, \theta) \equiv \{q' \in \mathcal{R}_+ \mid U(x, \frac{y}{w(\theta, q^f + q')}) = V(x, y, q^f, \theta)\} \tag{2}$$

that is increasing in x and y, and decreasing in q^f (namely, $\frac{dq^s}{dq^f} \in (-1, 0)$). The effect of θ on q^s is such that:

- if capability and input are strong technologic complements (say, there is a $w^*_{q\theta} > 0$, such that $w_{q\theta} > w^*_{q\theta}$), then $\frac{dq^s}{d\theta} > 0$;
- if capability and input are weak technologic complements or substitute (say $w_{q\theta} < w^*_{q\theta}$), then $\frac{dq^s}{d\theta} < 0$.

In the Appendix, we show that the SCP is always satisfied for opting-in households, namely

$$\frac{d}{d\theta}(\frac{dx}{dy} \mid_U) = \frac{\partial}{\partial\theta}(\frac{dx}{dy} \mid_U) < 0.$$

3.1.3 First best benchmark and mimicking behaviors

In first best, government observes exogenous capabilities of individuals and can implement optimal lump sum taxation. In this case, cash redistribution is superior to in-kind trans-

[4]Moreover, such in such a case the total effect of exogenous capability on individual productivity is negative and the very concept of high- and low- capability could be questioned.

[5]Such a choice is observed by the government, hence the most general tax schedule could depend also on it. However, at the optimum households with the same exogenous capability choose either to opt in or to privately supplement, therefore the case for a general tax schedule depending on individual choice to opt for the second-pillar provision happens only out of the tax equilibrium.

fers. Thus, in first best there is no role for the public provision of input, and without loss of generality we put $q^f = q^s = 0$. The first best optimization conditions imply non-distorting optimal taxation, namely

$$\frac{dx}{dy}\mid_V = -\frac{U_l}{U_c \cdot w} = 1$$

for all θ. Moreover, by individual optimization (1) also $\frac{w}{y \cdot w_q} = 1$, for all θ.

The first best allocation may be incentive-incompatible. However, as usual, poor households have no incentive to mimic rich households, while the reverse can happen, namely when redistribution is sufficiently large. In the following, we will assume that this is the case.

3.1.4 Policy regimes

In the considered setting, we have four possible of policy regimes depending on government policies and structural parameters (Greco, 2011):

PT in the pure taxation regime, the level of q^f and q^s are such that no individual is constrained by the first pillar and no individual opt for the second pillar public provision, hence $q^s < \min\{\underline{q}^s, \overline{q}^s\}$ (where $\underline{q}^s \equiv q^s(\underline{x}, \underline{y}, q^f, \underline{\theta})$ and $\overline{q}^s \equiv q^s(\overline{x}, \overline{y}, q^f, \overline{\theta})\}$);

INC in the inclusive regime, the level of q^f and q^s are such that all individuals are constrained by the first pillar or opt for the second pillar public provision, hence $q^s > \max\{\underline{q}^s, \overline{q}^s\}$;

DM we may have two discriminating regimes, depending on the type (high- or low-capability) of households with higher minimum public provision inducing them to opt for the second pillar scheme:

DML when $\underline{q}^s \leq q^s \leq \overline{q}^s$, low-capability households opt for the second-pillar provision while high-capability households opt out;

DMH when $\underline{q}^s \geq q^s \geq \overline{q}^s$, high-capability households opt for the second-pillar provision while low-capability households opt out.

4 Optimal Taxation and Public Provision

The government may choose the relevant policy regime, by setting appropriate tax schedule and public provision levels for the first and second pillar. However, the nature of the prevailing discriminating regime also depends on structural features such as individual preferences and technology. Passing from one policy regime to the other introduces, in the two-class setting, discontinuities in the structure of government objective and constraints, therefore we will firstly find optimal solutions within each policy regime, then we will analyze global optima.

4.1 Discriminating Regimes

In this section, we analyze optimal solutions in the discriminating regimes. First, we consider the case in which, given the first-pillar public provision and the optimal tax schedule, low-capability households opt for the second-pillar public provision, while high-capability ones opt out. In the following subsection, we will focus on the other case in which high-capability households opt for the second-pillar provision, while low-capability households opt out.

4.1.1 DML: low-capability households opt in

In this discriminating regime, the low-ability opt in, the high-ability opt out, thus the high-capability households mimicking low-capability are forced to opt for the second pillar public provision. Therefore the maximization problem of the government is

$$\max_{\{\underline{x},\underline{y},\overline{x},\overline{y},q^f,q^s\}} \lambda \cdot U(\underline{x}, \frac{\underline{y}}{w(\underline{\theta}, q^f + q^s)}) + (1-\lambda) \cdot V(\overline{x}, \overline{y}, q^f, \overline{\theta})$$

$$s.t. : \tag{3}$$

$$\lambda \cdot (\underline{y} - \underline{x}) + (1-\lambda) \cdot (\overline{y} - \overline{x}) - q^f - \lambda \cdot q^s \geq 0$$

$$V(\overline{x}, \overline{y}, q^f, \overline{\theta}) \geq U(\underline{x}, \frac{\underline{y}}{w(\overline{\theta}, q^f + q^s)})$$

$$q^s(\overline{x}, \overline{y}, q^f, \overline{\theta}) \geq q^s$$

$$q^s \geq q^s(\underline{x}, \underline{y}, q^f, \underline{\theta})$$

$$q^f \geq 0$$

$$q^s(\overline{x}, \overline{y}, q^f, \overline{\theta}) \geq 0$$

By the optimization conditions of the program (3) we can characterize the optimal taxation schedule and public provision scheme. Let us first abstract from corner solutions. In other terms, we consider that the heterogeneity between low- and high-capability households is sufficiently strong that lower and upper constraints on q^s and q^f do not bind. This insures that there is enough scope for the public provision program to discriminate between different classes, providing each of them with a different (total) level of publicly provided input, namely the rich will take only the first pillar provision, while the poor will benefit of the sum of first- and second-pillar publicly provided input. Thus, we have

Proposition 1 *Under the DML regime and assuming that the scope for the second-pillar public provision of input is sufficiently wide, the optimal discriminating policy,* $\{\underline{x}^*, \underline{y}^*, \overline{x}^*, \overline{y}^*, q^{f*}, q^{s*}\}$*, is such that:*

1. *the optimal public provision mix is equivalent to pure opting-out (i.e.,* $q^{f*} = 0$ *and* $q^{s*} > 0$*), and such that low-capability households are constrained to over-invest (respectively, under-invest) if exogenous capability and input are economic substitutes (respectively, complements);*

2. the optimal taxation schedule performs the usual shape as under the PT regime.

The proof of the Proposition is provided in the Appendix. Remark that the optimal public provision scheme can be a mixed one (where both the first- and the second-pillar provision are active), but any mixed scheme is equivalent (in terms of individual behaviors, government budget constraint, and social welfare) to a pure opting-out scheme where only the second pillar is active. The reason is that - in the considered setting - the first-pillar provision is equivalent to a cash transfer to everybody. Interestingly, the level of (second-pillar) publicly provided input depends on complementarity or substitutability of input and capability. As common in the literature on the public provision of private goods, this is justified by the impact of the public provision on the incentive constraint of the rich. The high-capability household trying to mimic the poor is forced to take the publicly provided input. Now, when input and capability are economic complements, the mimicker is hurt by any reduction of the publicly provided input; and conversely, when input and capability are economic substitutes, increasing the public provision beyond what low-capability households would choose on their own (given the second-best taxation schedule) is a means to transfer them more resources without any additional cost in terms of incentives.

Interestingly, the optimal taxation schedule is unaffected by the optimal structure of the public provision program. This last feature is affected when we consider corner solutions. If the scope for the second-pillar program is limited by the aim to discriminate between the rich (that should be opt out) and the poor (that should opt in), then the optimal taxation schedule has to partly offset the effect of constraints. Also, the optimal structure of the public provision program may be affected. As pointed out by the following propositions, the relevant corner solution depends on the economic complementarity or substitutability between input and capability.

Proposition 2 *Under the DML regime, if exogenous capability and input are economic* *substitutes and the scope for the second-pillar public provision is limited by self-selection constraints:*

1. *the optimal provision mix is pure topping up, and such that* $\overline{q}^s = \overline{q}^m = 0$*;*

2. *the optimal tax schedule subsidizes at the margin high-capability labor supply, and taxes at the margin (with the usual shape of tax wedge) low-capability labor supply.*

Again the proof is provided in the Appendix. First, when input and capability are economic substitutes the only possible corner solution is that the upper constraint on q^s binds. This is intuitive, given that in this case it is (second-best) optimal to overextend the public provision of input. We may have a corner solution because the heterogeneity between households is relatively small, hence it would be optimal to have a level of public provision which is beyond the minimum level inducing the high-capability households to opt in. In this case, government need to rely only on the first-pillar provision. Remark that, by the Proposition 2, this implies that the optimal publicly provided input is equal to the level of input that the rich households would choose on their own without any public provision (but with the same tax schedule). The optimal tax schedule is now affected. In particular, given that the optimal public provision is limited from above, an additional incentive is provided to high-capability labor supply.

We now consider the other case:

Proposition 3 *Under the DML regime, if exogenous capability and input are economic complements, and the scope for the second-pillar public provision is limited by self-selection constraints:*

1. *the optimal public provision mix is pure opting-out;*

2. *the optimal taxation schedule does not distort high-capability labor supply, and distorts low-capability labor supply with a marginal tax featured by an additional term with respect to the* PT *regime.*

When input and capability are complements, the public provision is a tool to harden the incentive constraint of the rich mimicking the poor. If heterogeneity of households is too small, the lower bound on the second-pillar public provision binds; in other terms, the optimal publicly provided input would be lower than the minimum level needed to convince low-capability households to opt in. Therefore, the marginal tax wedge on the poor involves an additional term to discourage mimicking.

We now deepen our investigation by a numerical characterization of the considered setting. [INCOMPLETE HERE]. We consider a standard functional form of the utility function (Heatcote *et al.*, 2009):

$$U(c,l) = \frac{c^{1-\rho}-1}{1-\rho} - \frac{l^{1+k}}{1+k}$$

The parameter ρ represents the coefficient of relative risk-aversion (as baseline case, we take $\rho = 3$) and $\frac{1}{k}$ is the Frisch elasticity of labor supply. In particular, we set $k = 2$ in order to account for the standard compensated elasticity of labor estimated in literature, 0.33 (CITATION HERE). Consistently with the literature on Mincer's earning functions (CITATION HERE), we use the following Cobb-Douglas specification of individual earning function

$$w(\theta, q^f + q^m) = \theta^\alpha \cdot (q^f + q^m)^\beta$$

Using a log-transformation, it is easy to see that α and β represent the estimated coefficients of a Mincer's regression with regressors θ and q, respectively. However, we also check the robustness of our results using a linear earning function (i.e., with θ and q technologic substitutes).

Table 1 reports our benchmark parametrization.

Table 1: Benchmark Case

Parameter	Value
$\underline{\theta}$	1
$\overline{\theta}$	3
α	0.5
β	0.5
α	0.2
k	2
ρ	3

In the benchmark specification, we assume that θ and q have the same weight.[6] In the benchmark case, the discriminating regime is DML (i.e., low-capability households opt in for lower second-pillar provision). Table 2 summarizes the results featuring the optimal discriminating policy.

Table 2: Optimal Policies (DML regime)

Instruments	Benchmark	$\alpha = 0.4$, $\beta = 0.6$	$k = 6$	$\overline{\theta}= 5$	$\lambda= 0.55$
$\underline{y}$	0.380	0.229	0.562	0.099	1.176
$\underline{x}$	0.614	0.571	0.605	0.724	1.481
$\overline{y}$	2.898	3.810	2.270	3.872	1.823
$\overline{x}$	2.320	3.281	1.801	3.168	0.701
q^f	0.155	0.050	0.145	0.030	0.235
q^s	0.035	0.087	0.136	0.020	0.014
Welfare	-0.960	-1.124	-0.975	-0.470	-3.833

As expected both taxation and public provision are used as redistribution tools. In the benchmark case, θ and q are economic complements and as expected the optimal tax

[6]Heckman *et al.* (2006) show that inherited abilities and family background (our exogenous capability) have similar weight than schooling in the determination of individual wage.

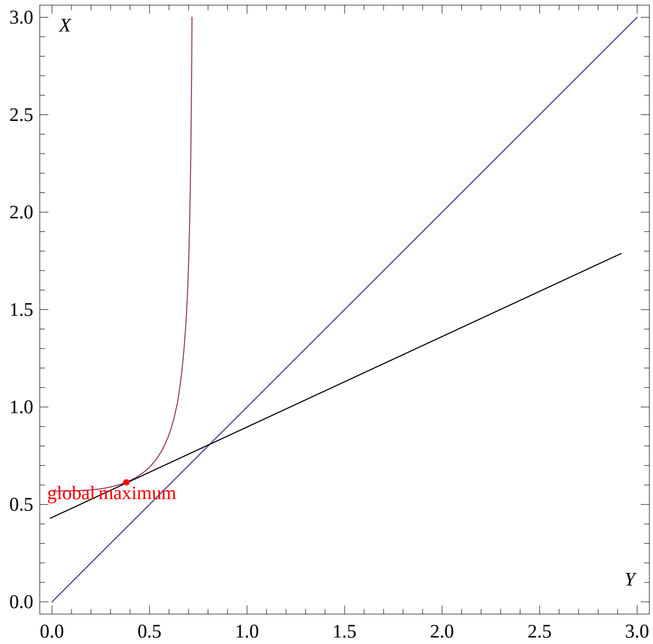

Figure 1: DML - interior solutions: Low-capability household

schedule does not distort high-capability households (Figure 1), while the marginal tax on low-capability households is positive (Figure 2).

We now consider an alternative specification of the earning function, $w(\theta, q^f + q^m) = \alpha \cdot \theta + \beta \cdot (q^f + q^m)$. In this case, capability and input are economic substitutes. With the benchmark parametrization of the Table 1, the optimal discriminating regime is $\{\underline{y}, \underline{x}, \overline{y}, \overline{x}, q^f, q^s\} = \{0.364, 0.791, 1.721, 1.294, 0, 0.0002\}$. Consistently with Proposition 1, we find that the optimal public provision scheme is pure opting out, given that the scope for the second pillar is sufficiently wide (i.e., $q^f = 0$, and $q^s(\overline{x}, \overline{y}, q^f, \overline{\theta}) = 0.0006 > q^{s*} = 0.0002$.

Keeping the latter specification, we assume now that $\overline{\theta} = 2$. With this change, there is not enough scope for the second pillar public provision, hence both pillars are active and the optimal tax schedule for high-capability households involves a marginal subsidy (Figure 4).

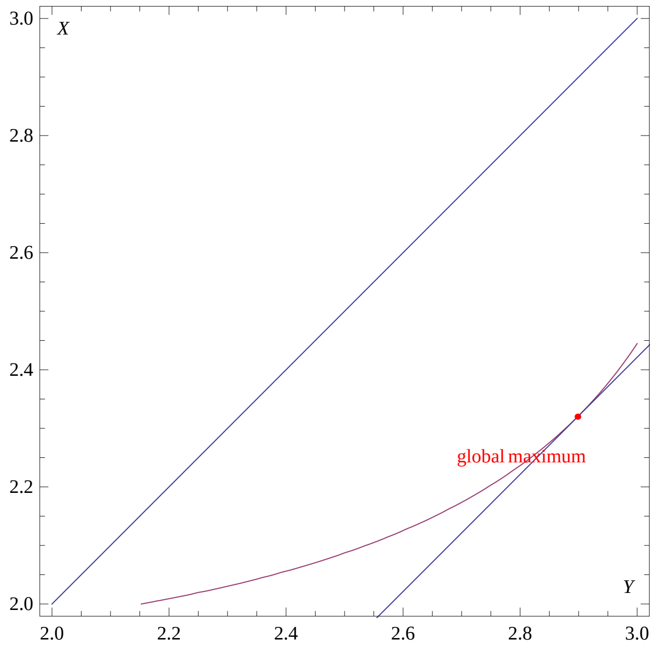

Figure 2: DML - interior solutions: High-capability household

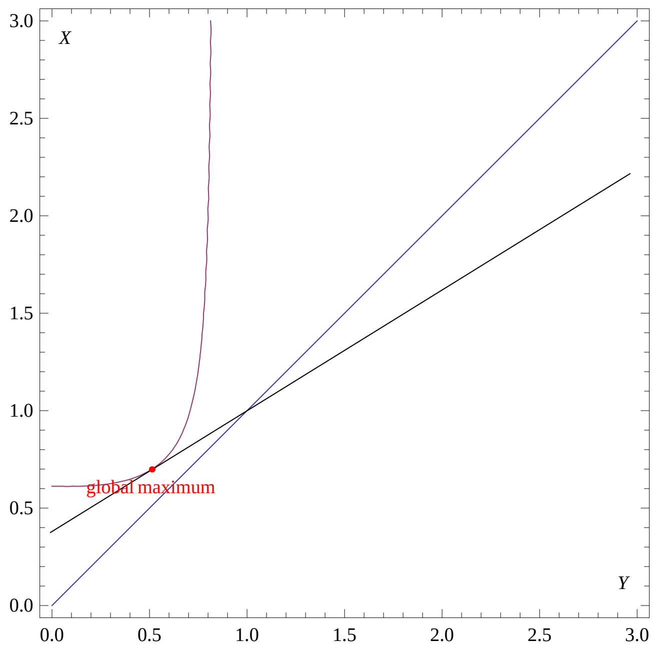

Figure 3: DML - upper corner solution: Low-capability household

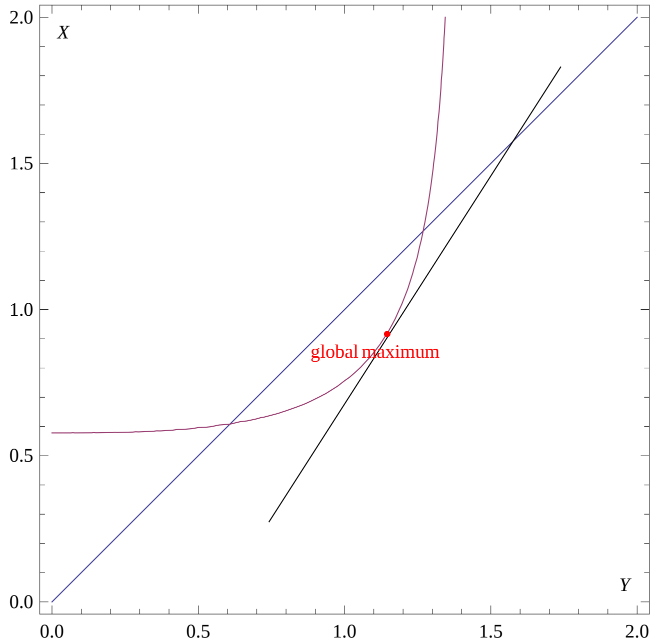

Figure 4: DML- upper corner solution: High-capability household

4.1.2 DMH: high-capability households opt in

We now consider the other discriminating case, where high-capability households opt-in for lower level of q^s than low-capability. The maximization problem of the government is

$$\begin{aligned}
\max_{\{\underline{x},\underline{y},\overline{x},\overline{y},q^f,q^s\}} \lambda \cdot V(\underline{x},\underline{y},q^f,\underline{\theta}) + (1-\lambda)\cdot U(\overline{x},\frac{\overline{y}}{w(\overline{\theta},q^f+q^s)})& \\
s.t.:& \qquad (4) \\
\lambda\cdot(\underline{y}-\underline{x}) + (1-\lambda)\cdot(\overline{y}-\overline{x}) - q^f - (1-\lambda)\cdot q^s \geq 0& \\
U(\overline{x},\frac{\overline{y}}{w(\overline{\theta},q^f+q^s)}) \geq V(\underline{x},\underline{y},q^f,\overline{\theta})& \\
q^s(\underline{x},\underline{y},q^f,\underline{\theta}) \geq q^s& \\
q^s \geq q^s(\overline{x},\overline{y},q^f,\overline{\theta})& \\
q^f \geq 0& \\
q^s(\underline{x},\underline{y},q^f,\underline{\theta}) \geq 0&
\end{aligned}$$

By the first order conditions of (4), we get the following

Proposition 4 *Under the DMH regime, the optimal discriminating policy replicates the optimal taxation policy under the PT regime. In particular, public provision is irrelevant.*

The proof is provided in the Appendix. Interestingly, when the rich enter the public provision scheme for lower levels of publicly provided input, then the public provision program does not add anything to pure taxation as redistribution tool. Given that the minimum level of input that induces households to opt for the second-pillar public provision increases in gross and net incomes, to observe a DML regime we need that the degree of technologic complementarity is not too strong.

4.2 Welfare Analysis

Which policy (among the regime-specific optima) is globally optimal? In this section, we first provide a qualitative answer to this question, then we characterize the optimality of alternative regime by means of numerical specifications of our general setting.

Let us first consider the DML regime. In such a case, the optimal discriminating policy replicates the pure optimal taxation, therefore the public provision policy is irrelevant. Conversely, when a DMH regime is relevant, the optimal discriminating policy never replicates pure optimal taxation. Thus, when the poor opt in for lower level of publicly provided input, the public provision is always welfare-improving with respect to pure optimal taxation.

Let us now consider the comparison with the inclusive regime [INCOMPLETE HERE]

5 Multi-class Economy

In this section, we consider an extended version of our model with an infinite number of individual-capability classes, namely $\theta \in [\underline{\theta}, \overline{\theta}]$, with probability and density functions $F(\theta)$ and $f(\theta)$, respectively. As in the two-classes case, we need to discriminate between two situations: $\frac{dq^s}{d\theta} = q^s_\theta > 0$ or $\frac{dq^s}{d\theta} = q^s_\theta < 0$[7]. In the first case, for any level of first- and second-pillar public provision, we can find an household with capability $\hat{\theta} \in [\underline{\theta}, \overline{\theta}]$ such that households with $\theta \leq \hat{\theta}$ opt for the (second-pillar) public provision, while households with $\theta > \hat{\theta}$ opt out. In particular:

$$\hat{\theta} \equiv \{\theta \mid U(x(\theta), \frac{y(\theta)}{w(\theta, q^f + q^s)} = V(x(\theta), y(\theta), q^f, \theta)\}$$

Under this specification, the optimal control problem of the government can be generally represented as follows. The government maximizes the sum of households' utilities

$$\max \int_{\underline{\theta}}^{\hat{\theta}} u(\theta) \cdot dF(\theta) + \int_{\hat{\theta}}^{\overline{\theta}} v(\theta) \cdot dF(\theta)$$

(where $u(\theta) = U(x(\theta), \frac{y(\theta)}{w(\theta, q^f+q^s)})$ and $v(\theta) = V(x(\theta), y(\theta), q^f, \theta)$, and by definition of $\hat{\theta}$, also $u(\hat{\theta}) = v(\hat{\theta})$), under the public budget constraint

$$\int_{\underline{\theta}}^{\overline{\theta}} [y(\theta) - x(\theta)] \cdot dF(\theta) - q^f - q^s \cdot F(\hat{\theta}) \geq 0,$$

the incentive constraint of opting-in households

$$u(\theta) \geq U(x(\theta'), \frac{y(\theta')}{w(\theta, q^f + q^s)})$$

for all $\theta' \neq \theta$ and all $\theta \leq \hat{\theta}$, and the incentive constraint of opting-out households

$$v(\theta) \geq V(x(\theta'), y(\theta'), q^f, \theta))$$

for all $\theta' \neq \theta$ and all $\theta > \hat{\theta}$. Let us remark that, given the utility level, $u(\theta)$ (or $v(\theta)$), and the gross income $y(\theta)$, the consumption level $x(\theta)$ is determined by definition of $u(.)$ (or

[7]Remark that $\frac{dq^s}{d\theta} = q^s_x \cdot x' + q^s_y \cdot y' + q^s_\theta = q^s_\theta$, given that by individual households' optimization $q^s_x \cdot x' + q^s_y \cdot y' = 0$

$v(.)$), hence we can substitute $x(\theta) = x(y(\theta), u(\theta), q^f, q^s, \theta)$, with $x(y(\theta), u(\theta)q^f, q^s, \theta) \equiv \{x \mid u(\theta) = U(x(\theta), \frac{y(\theta)}{w(\theta, q^f + q^s)})\}$.

As usual in optimal control modeling of taxation, $u(\theta)$, $v(\theta)$, $y(\theta)$, and

$$R(\theta) = \int_{\underline{\theta}}^{\theta} [y(\tilde{\theta}) - x(\tilde{\theta}) - q^f - q^s \cdot F(\hat{\theta})] \cdot dF(\tilde{\theta})$$

are treated as state variables, with $\mu(\theta)$, $\gamma(\theta)$, and $\lambda(\theta)$ corresponding co-state variables. The control variables are $\eta(\theta)$ the change of gross income with respect to households' capability, q^f and q^s.

With this specification the government control problem can be written as

$$
\begin{aligned}
\max \int_{\underline{\theta}}^{\hat{\theta}} u(\theta) \cdot dF(\theta) + \int_{\hat{\theta}}^{\overline{\theta}} v(\theta) \cdot dF(\theta) \\
s.t. \\
\frac{dR}{d\theta} = [y(\theta) - x(y(\theta), u(\theta), q^f, q^s, \theta) - q^f - q^s \cdot F(\hat{\theta})] \cdot f(\theta) \quad (\lambda(\theta)) \quad \forall \theta \leq \hat{\theta} \\
\frac{dR}{d\theta} = [y(\theta) - x(y(\theta), v(\theta), q^f, \theta) - q^f - q^s \cdot F(\hat{\theta})] \cdot f(\theta) \quad (\lambda(\theta)) \quad \forall \theta > \hat{\theta} \\
\frac{du}{d\theta} = -\frac{U_l^s}{w^{s2}} \cdot w_\theta^s \quad (\mu(\theta)) \quad \forall \theta \leq \hat{\theta} \\
\frac{dv}{d\theta} = -\frac{U_l^m}{w^{m2}} \cdot w_\theta^s \quad (\mu(\theta)) \quad \forall \theta > \hat{\theta} \\
\frac{dy}{d\theta} = \eta(\theta) \quad (\gamma(\theta)) \\
\eta(\theta) \geq 0 \quad (\sigma(\theta)) \\
q^f \cdot f(\theta) \geq 0 \quad (\phi(\theta)) \\
q^s \cdot f(\theta) \geq 0 \quad (\varphi(\theta)) \\
R(\underline{\theta}) = 0 \\
R(\overline{\theta}) = 0 \\
\mu(\underline{\theta}) = 0 \\
\nu(\overline{\theta}) = 0 \\
\gamma(\underline{\theta}) = 0 \\
\gamma(\overline{\theta}) = 0
\end{aligned}
\tag{5}
$$

Thus the Hamiltonian for the optimization program is

$$
\begin{aligned}
H = u(\theta) \cdot f(\theta) + \lambda(\theta) \cdot [y(\theta) - x(y(\theta), u(\theta), q^f, q^s, \theta) - q^f - q^s \cdot F(\hat{\theta})] \cdot f(\theta) - \mu(\theta) \cdot \frac{U_l^s}{w^{s2}} \cdot w_\theta^s + \\
+ \gamma(\theta) \cdot \eta(\theta) + \sigma(\theta) \cdot \eta(\theta) + \phi(\theta) \cdot f(\theta) \cdot q^f + \varphi(\theta) \cdot f(\theta) \cdot q^s
\end{aligned}
\tag{6}
$$

for all $\theta \leq \hat{\theta}$, and

$$H = v(\theta) \cdot f(\theta) + \lambda(\theta) \cdot [y(\theta) - x(y(\theta), v(\theta), q^f, \theta) - q^f - q^s \cdot F(\hat{\theta})] \cdot f(\theta) - \nu(\theta) \cdot \frac{U_l^m}{w^{m2}} \cdot w_\theta^s +$$
$$+\gamma(\theta) \cdot \eta(\theta) + \sigma(\theta) \cdot \eta(\theta) + \phi(\theta) \cdot f(\theta) \cdot q^f + \varphi(\theta) \cdot f(\theta) \cdot q^s \tag{7}$$

for all $\theta > \hat{\theta}$.

The maximum principle conditions are

$$\frac{\partial H}{\partial \eta} = \gamma(\theta) + \sigma(\theta) = 0 \quad \sigma(\theta) \cdot \eta(\theta) = 0$$
$$\frac{\partial H}{\partial q^f} = \lambda(\theta) \cdot (-\frac{dx}{dq^f} - 1 - q^s \cdot f(\hat{\theta}) \cdot \frac{d\hat{\theta}}{dq^f}) \cdot f(\theta) - \mu(\theta) \cdot (\frac{w_\theta^s}{w^{s2}} \cdot (U_{cl}^s \cdot \frac{dx}{dq^f} - \frac{U_{ll}^s}{w^{s2}} \cdot w_q^s) +$$
$$+U_l^s \cdot (-\frac{2 \cdot w_q^s \cdot w_\theta^s}{w^{s3}} + \frac{w_{q\theta}^s}{w^{s2}})) + \phi(\theta) \cdot f(\theta) = 0 \quad \phi(\theta) \cdot f(\theta) \cdot q^f = 0$$
$$\frac{\partial H}{\partial q^s} = \lambda(\theta) \cdot (-\frac{dx}{dq^s} - F(\hat{\theta}) - q^s \cdot f(\hat{\theta}) \cdot \frac{d\hat{\theta}}{dq^s}) \cdot f(\theta) - \mu(\theta) \cdot (\frac{w_\theta^s}{w^{s2}} \cdot (U_{cl}^s \cdot \frac{dx}{dq^s} - \frac{U_{ll}^s}{w^{s2}} \cdot w_q^s) +$$
$$+U_l^s \cdot (-\frac{2 \cdot w_q^s \cdot w_\theta^s}{w^{s3}} + \frac{w_{q\theta}^s}{w^{s2}})) + \varphi(\theta) \cdot f(\theta) = 0 \quad \varphi(\theta) \cdot f(\theta) \cdot q^s = 0$$
$$\frac{dR}{d\theta} = [y(\theta) - x(y(\theta), u(\theta), q^f, q^s, \theta) - q^f - q^s \cdot F(\hat{\theta})] \cdot f(\theta)$$
$$\frac{du}{d\theta} = -\frac{U_l^s}{w^{s2}} \cdot w_\theta^s$$
$$\frac{dy}{d\theta} = \eta(\theta)$$
$$\frac{d\lambda}{d\theta} = -\frac{\partial H}{\partial R} = 0$$
$$\frac{d\mu}{d\theta} = -\frac{\partial H}{\partial u} = -f(\theta) - \lambda(\theta) \cdot (-\frac{dx}{du} - q^s \cdot f(\hat{\theta}) \cdot \frac{d\hat{\theta}}{dx} \cdot \frac{dx}{du}) \cdot f(\theta) + \mu(\theta) \cdot \frac{U_{cl}^s}{w^{s2}} \cdot w_\theta^s \cdot \frac{dx}{du}$$
$$\frac{d\gamma}{d\theta} = -\frac{\partial H}{\partial y} = -\lambda(\theta) \cdot (1 - \frac{dx}{dy} - q^s \cdot f(\hat{\theta}) \cdot \frac{d\hat{\theta}}{dx} \cdot \frac{dx}{dy}) \cdot f(\theta) + \mu(\theta) \cdot (\frac{U_{ll}^s}{w^{s2}} \cdot w_\theta^s + \frac{U_{cl}^s}{w^{s2}} \cdot w_\theta^s \cdot \frac{dx}{dy})$$

for all $\theta \leq \hat{\theta}$, and

$$\frac{\partial H}{\partial \eta} = \gamma(\theta) + \sigma(\theta) = 0 \quad \sigma(\theta) \cdot \eta(\theta) = 0$$

$$\frac{\partial H}{\partial q^f} = \lambda(\theta) \cdot (-\frac{dx}{dq^f} - 1 - q^s \cdot f(\hat{\theta}) \cdot \frac{d\hat{\theta}}{dq^f}) \cdot f(\theta) - \mu(\theta) \cdot (\frac{w_\theta^m}{w^{m2}} \cdot (U_{cl}^m \cdot \frac{dx}{dq^f} - \frac{U_{ll}^m}{w^{m2}} \cdot w_q^m) +$$

$$+U_l^m \cdot (-\frac{2 \cdot w_q^m \cdot w_\theta^m}{w^{m3}} + \frac{w_{q\theta}^m}{w^{m2}})) + \phi(\theta) \cdot f(\theta) = 0 \quad \phi(\theta) \cdot f(\theta) \cdot q^f = 0$$

$$\frac{\partial H}{\partial q^s} = \lambda(\theta) \cdot (-F(\hat{\theta}) - q^s \cdot f(\hat{\theta}) \cdot \frac{d\hat{\theta}}{dq^s}) \cdot f(\theta) + \varphi(\theta) \cdot f(\theta) = 0 \quad \varphi(\theta) \cdot f(\theta) \cdot q^s = 0$$

$$\frac{dR}{d\theta} = [y(\theta) - x(y(\theta), v(\theta), q^f, \theta) - q^f - q^s \cdot F(\hat{\theta})] \cdot f(\theta)$$

$$\frac{du}{d\theta} = -\frac{U_l^m}{w^{m2}} \cdot w_\theta^m$$

$$\frac{dy}{d\theta} = \eta(\theta)$$

$$\frac{d\lambda}{d\theta} = -\frac{\partial H}{\partial R} = 0$$

$$\frac{d\mu}{d\theta} = -\frac{\partial H}{\partial v} = -f(\theta) - \lambda(\theta) \cdot (-\frac{dx}{dv} - q^s \cdot f(\hat{\theta}) \cdot \frac{d\hat{\theta}}{dx} \cdot \frac{dx}{dv}) \cdot f(\theta) + \mu(\theta) \cdot \frac{U_{cl}^m}{w^{m2}} \cdot w_\theta^m \cdot \frac{dx}{dv}$$

$$\frac{d\gamma}{d\theta} = -\frac{\partial H}{\partial y} = -\lambda(\theta) \cdot (1 - \frac{dx}{dy} - q^s \cdot f(\hat{\theta}) \cdot \frac{d\hat{\theta}}{dx} \cdot \frac{dx}{dy}) \cdot f(\theta) + \mu(\theta) \cdot (\frac{U_{ll}^m}{w^{m2}} \cdot w_\theta^m + \frac{U_{cl}^m}{w^{m2}} \cdot w_\theta^m \cdot \frac{dx}{dy})$$

for all $\theta > \hat{\theta}$.

Remark that, by these conditions $\lambda(\theta) = \lambda$ for all θ, hence it is possible show that if $\frac{d\hat{\theta}}{dq^s} > 0$, then necessarily the optimal $q^s = 0$, hence $\hat{\theta} = 0$. Thus, a necessary condition for q^s to optimally be positive is that $\frac{d\hat{\theta}}{dq^s} < 0$.

6 Conclusions

[INCOMPLETE HERE]

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1 Technical Appendix

1.1 Household's input demand

Let us consider the behavior of households opting out of the second pillar. (To save space we omit household's index, i). Taking as given the government's tax policy and the first-pillar public provision of investment (q^f) as well as the generic household's choice about labor supply (hence, the net - x - and gross - y - incomes), the optimal private investment of the household is determined by the program

$$\max_{q^m} U(x - q^m, \frac{y}{w(\theta, q^f + q^m)}) \quad s.t. \quad q^m \geq 0$$

The first order condition is

$$\frac{dU}{dq^m} = \phi = -U_c - U_l \cdot \frac{y}{w^2} \cdot w_q = 0 \tag{8}$$

By concavity of $U(.,.)$ (hence, $U_{cc} + 2 \cdot U_{cl} \cdot \frac{y}{w^2} \cdot w_q + U_{ll} \cdot (\frac{y}{w^2} \cdot w_q)^2 < 0$) and $w(.,.)$ (hence, $w_{qq} \leq 0$), the second order condition with respect to q^m is satisfied

$$\frac{d^2U}{dq^{m2}} = \phi_{q^m} = U_{cc} + 2 \cdot U_{cl} \cdot \frac{y}{w^2} \cdot w_q + U_{ll} \cdot (\frac{y}{w^2} \cdot w_q)^2 + \\ + U_l \cdot (\frac{2 \cdot y}{w^3} \cdot w_q^2 - \frac{y}{w^2} \cdot w_{qq}) < 0 \tag{9}$$

1.1.1 Comparative statics

We now characterize the shape of $q^m(x, y, q^f, \theta)$. Applying the implicit function theorem to the first order condition with respect to q^m, we know that

$$\frac{dq^m}{dz} = -\frac{\phi_z}{\phi_{q^m}}$$

where $z \in \{x, y, q^f, \theta\}$. By weak concavity of $w(.,.)$ in q, we know that $\phi_{q^m} < 0$, thus the effect of z on $q^m(x, y, q^f, \theta)$ depends on the differential of the first order condition, ϕ_z. We summarize all results in the following Lemmas.

Lemma 5 *Private input demand increases in net income less than one-to-one:* $\frac{dq^m}{dx} \in (0, 1)$.

Proof. $\frac{dq^m}{dx} > 0$ if and only if $\phi_x = -U_{cc} - U_{lc} \cdot \frac{y}{w^2} \cdot w_q > 0$. By (8), $-\frac{U_c}{U_l} = \frac{y}{w^2} \cdot w_q$, thus $\phi_x = -U_{cc} + \frac{U_c}{U_l} \cdot U_{lc} > 0$ if and only if leisure is a normal good. Moreover, $\frac{dq^m}{dx} < 1$ if $\phi_x < -\phi_{q^m}$ or $(-U_{cl} + \frac{U_c}{U_l} \cdot U_{ll}) \cdot \frac{y}{w^2} \cdot w_q - U_l \cdot (2 \cdot \frac{y}{w^3} \cdot w_q^2 - \frac{y}{w^2} \cdot w_{qq}) > 0$, that is satisfied if consumption is a normal good (hence, $-U_{cl} + \frac{U_c}{U_l} \cdot U_{ll} > 0$). ■

Lemma 6 *Private input demand increases in gross income less than the marginal rate of substitution between net and gross income:* $\frac{dq^m}{dy} \in (0, \frac{dx}{dy} |_V)$.

Proof. $\frac{dq^m}{dy} > 0$ if $\phi_y = -U_{cl} \cdot \frac{1}{w} - U_{ll} \cdot \frac{y}{w^3} \cdot w_q - U_l \cdot \frac{w_q}{w^2} > 0$. By (8), $-\frac{U_c}{U_l} = \frac{y}{w^2} \cdot w_q$, thus $\phi_y = \frac{1}{w} \cdot \left(-U_{cl} + \frac{U_c}{U_l} \cdot U_{ll} - U_l \cdot \frac{w_q}{w}\right) > 0$ that is true if consumption is a normal good (hence, $-U_{cl} + \frac{U_c}{U_l} \cdot U_{ll} > 0$). Moreover, $\frac{dq^m}{dy} = \frac{w}{y \cdot w_q} \cdot \alpha_y$ or - by the first order conditions for q^m and consumption-labor choices - $\frac{dq^m}{dy} = \frac{dx}{dy} |_V \cdot \alpha_y$, where

$$\alpha_y = \frac{U_{cl} \cdot \frac{y}{w^2} \cdot w_q + U_{ll} \cdot (\frac{y}{w^2} \cdot w_q)^2 + U_l \cdot \frac{y}{w^3} \cdot w_q^2}{U_{cc} + 2 \cdot U_{cl} \cdot \frac{y}{w^2} \cdot w_q + U_{ll} \cdot (\frac{y}{w^2} \cdot w_q)^2 + U_l \cdot (2 \cdot \frac{y}{w^3} \cdot w_q^2 - \frac{y}{w^2} \cdot w_{qq})} < 1$$

if $U_{cc} + U_{cl} \cdot \frac{y}{w^2} \cdot w_q + U_l \cdot (\frac{y}{w^3} \cdot w_q^2 - \frac{y}{w^2} \cdot w_{qq}) < 0$, that - by the first order condition - is satisfied if leisure is a normal good (hence, $U_{cc} - \frac{U_c}{U_l} \cdot U_{cl} < 0$) and $w(.,.)$ is concave in q. ■

Lemma 7 *The first-pillar public provision crowds partially out private input demand:* $\frac{dq^m}{dq^f} \in (-1, 0)$.

Proof. $\frac{dq^m}{dq^f} < 0$ if $\phi_{q^f} = \frac{y}{w^2} \cdot w_q \cdot (U_{cl} + \frac{y}{w^2} \cdot w_q \cdot U_{ll}) + U_l \cdot \left(2 \cdot \frac{y}{w^3} \cdot w_q^2 - \frac{y}{w^2} \cdot w_{qq}\right) < 0$. By (8), $\phi_{q^f} = \frac{y}{w^2} \cdot w_q \cdot (U_{cl} - \frac{U_c}{U_l} \cdot U_{ll}) + U_l \cdot \left(2 \cdot \frac{y}{w^3} \cdot w_q^2 - \frac{y}{w^2} \cdot w_{qq}\right) < 0$, that is true if consumption is a normal good (hence, $U_{cl} - \frac{U_c}{U_l} \cdot U_{ll} < 0$). Moreover, $\frac{dq^m}{dq^f} > -1$ if $\phi_{q^f} > \phi_{q^m}$ or $U_{cc} + \frac{y}{w^2} \cdot w_q \cdot U_{cl} < 0$, that is satisfied if leisure is a normal good (hence, $U_{cc} - \frac{U_c}{U_l} \cdot U_{cl} < 0$). ■

Let us remark that, for any given level of household's wage (e.g., w'), the marginal rate of technical substitution between input and capability that keeps constant wage is given by $\frac{dq}{d\theta} = -\frac{w_\theta}{w_q} \mid_{w'} < 0$.

Lemma 8 *Private input is an economic*

- *complement of capability (i.e., $\frac{dq^m}{d\theta} > 0$), if input and capability are strong technologic complements (i.e., $w_{q\theta} > w_{q\theta}^{+} > 0$);*
- *strong substitute for capability (i.e., $\frac{dq^m}{d\theta} < -\frac{w_\theta}{w_q}$), if input and capability are strong technologic substitutes (i.e., $w_{q\theta} < w_{q\theta}^{-} < 0$);*
- *substitute for capability (i.e., $\frac{dq^m}{d\theta} \in (-\frac{w_\theta}{w_q}, 0)$), if capability and input are weak technologic complements or substitutes (i.e., $w_{q\theta} \in (w_{q\theta}^{-}, w_{q\theta}^{+})$).*

Proof.

$$\phi_\theta = \frac{y}{w^2} \cdot w_\theta \cdot (U_{cl} + \frac{y}{w^2} \cdot w_q \cdot U_{ll}) + U_l \cdot \left(2 \cdot \frac{y}{w^3} \cdot w_q \cdot w_\theta - \frac{y}{w^2} \cdot w_{q\theta}\right) \tag{10}$$

By (8), $U_{cl} + \frac{y}{w^2} \cdot w_q \cdot U_{ll} = U_{cl} - \frac{U_c}{U_l} \cdot U_{ll} < 0$ if consumption is normal. (10) is negative (or positive) if and only if $w_{q\theta} < w_{q\theta}^{+}$ (or $w_{q\theta} > w_{q\theta}^{+}$), where

$$w_{q\theta}^{+} \equiv 2 \cdot \frac{w_q \cdot w_\theta}{w} + \frac{U_{cl} - \frac{U_c}{U_l} \cdot U_{ll}}{U_l} > 0. \tag{11}$$

Moreover, $\frac{dq^m}{d\theta} = -\frac{w_\theta}{w_q} \cdot \alpha_\theta$, where

$$\alpha_\theta \equiv \frac{U_{cl} \cdot \frac{y}{w^2} \cdot w_q + U_{ll} \cdot (\frac{y}{w^2} \cdot w_q)^2 + U_l \cdot (2 \cdot \frac{y}{w^3} \cdot w_q^2 - \frac{y}{w^2} \cdot \frac{w_q}{w_\theta} \cdot w_{q\theta})}{U_{cc} + 2 \cdot U_{cl} \cdot \frac{y}{w^2} \cdot w_q + U_{ll} \cdot (\frac{y}{w^2} \cdot w_q)^2 + U_l \cdot (2 \cdot \frac{y}{w^3} \cdot w_q^2 - \frac{y}{w^2} \cdot w_{qq})} < 1$$

if $U_{cc} + U_l \cdot \frac{y}{w^2} \cdot (\frac{w_q}{w_\theta} \cdot w_{q\theta} - w_{qq}) < 0$ or $w_{q\theta} > w^-_{q\theta}$ where

$$w^-_{q\theta} = \frac{w_\theta}{w_q} \cdot (-\frac{w^2}{y} \cdot \frac{U_{cc}}{U_l} + w_{qq}) < 0. \tag{12}$$

■

1.1.2 Single Crossing Property

The effect of θ on the marginal rate of substitution between net and gross income depends, in this setting, also on the reaction of q^m to such a change (Boadway and Marchand, 1995).

Lemma 9 *The single crossing property*

$$\frac{d\frac{dx}{dy}\mid_V}{d\theta} = \frac{\partial\frac{dx}{dy}\mid_V}{\partial\theta} + \frac{\partial\frac{dx}{dy}\mid_V}{\partial q} \cdot \frac{dq}{d\theta} < 0 \tag{13}$$

is satisfied if and only if $w_{q\theta} > w^\star_{q\theta}$. *Moreover, the single crossing property is violated whenever* $w_{q\theta} < w^-_{q\theta}$.

Proof. Remark that

$$\frac{\partial\frac{dx}{dy}\mid_V}{\partial\theta} = -\frac{w_\theta}{w_q} \cdot \frac{U_l}{U_c^2 \cdot w} \cdot (-\frac{U_c}{U_l} \cdot U_{cl} + \frac{U_c^2}{U_l^2} \cdot U_{ll} - U_c \cdot \frac{w_q}{w}) < 0$$

and

$$\frac{\partial\frac{dx}{dy}\mid_V}{\partial q} = -\frac{U_l}{U_c^2 \cdot w} \cdot (U_{cc} - 2 \cdot \frac{U_c}{U_l} \cdot U_{cl} + \frac{U_c^2}{U_l^2} \cdot U_{ll} - U_c \cdot \frac{w_q}{w}) < 0$$

then, (13) is satisfied if and only if

$$\frac{dq}{d\theta} > -\frac{\frac{\partial\frac{dx}{dy}|_V}{\partial\theta}}{\frac{\partial\frac{dx}{dy}|_V}{\partial q}}$$

that is equivalently written as $\beta > \alpha_\theta$, where

$$\beta \equiv \frac{-\frac{U_c}{U_l} \cdot U_{cl} + \frac{U_c^2}{U_l^2} \cdot U_{ll} - U_c \cdot \frac{w_q}{w}}{U_{cc} - 2 \cdot \frac{U_c}{U_l} \cdot U_{cl} + \frac{U_c^2}{U_l^2} \cdot U_{ll} - U_c \cdot \frac{w_q}{w}} \in (0,1)$$

under the assumption that c is normal and $U(.,.)$ is strictly concave), and boils down to

$$w_{q\theta} > w_{q\theta}^{\star}$$

with

$$w_{q\theta}^{\star} \equiv \frac{w_q \cdot w_\theta}{w} - \beta \cdot (\frac{w_q}{w} - w_{qq})$$

Remark that when $w_{q\theta} < w_{q\theta}^{-}$, then $\alpha_\theta \geq 1$, hence $\beta > \alpha_\theta$ - and (13) - is violated. ■

1.2 Minimum second-pillar provision

The minimum second-pillar provision inducing the household with capability θ to opt in is

$$q^s(x, y, q^f, \theta) \equiv \{q' \in \mathcal{R}_+ \mid U(x, \frac{y}{w(\theta, q^f + q')}) = V(x, y, q^f, \theta)\} \tag{14}$$

Let $U^s \equiv U(x, \frac{y}{w(\theta, q^f + q')})$ and $U^m \equiv \max_{q^m} U(x - q^m, \frac{y}{w(\theta, q^f + q^m)})$. Thus, as far as $q^m(x, y, q^f, \theta) > 0$, $U_c^s < U_c^m$, $|U_l^s| > |U_l^m|$, and $q^m(x, y, q^f, \theta) > q^s(x, y, q^f, \theta) > 0$ (hence, $w^m > w^s$, where $w^s \equiv w(\theta, q^f + q^s)$ and $w^m \equiv w(\theta, q^f + q^m(x, y, q^f, \theta))$). Moreover, in this case, changes in net and gross incomes affect both sides of (14). Thus,

$$\frac{dq^s}{dx} = (\frac{U_c^m}{U_c^s} - 1) \cdot \frac{\frac{w^s}{y \cdot w_q^s}}{\frac{dx}{dy}\mid_{U(x^s, \frac{y}{w^s})}} > 0$$

and

$$\frac{dq^s}{dy} = \frac{w^s}{y \cdot w_q^s} \cdot (1 - \frac{\frac{U_l^m}{w^m}}{\frac{U_l^s}{w^s}}) > 0$$

Given that $\eta \equiv \frac{U_l^m \cdot \frac{y}{w^m}}{U_l^s \cdot \frac{y}{w^s}} < 1$,

$$\frac{dq^s}{dq^f} = -(1 - \eta \cdot \frac{\frac{w_q^m}{w^m}}{\frac{w_q^s}{w^s}}) \in (-1, 0)$$

Given η, the sign of

$$\frac{dq^s}{d\theta} = -\frac{w_\theta^s}{w_q^s} \cdot (1 - \eta \cdot \frac{\frac{w_\theta^m}{w^m}}{\frac{w_\theta^s}{w^s}})$$

depends on the technical complementarity/substitutability between q and θ, namely $\frac{dq^s}{d\theta} < 0$ if and only if $1 - \eta \cdot \frac{\frac{w_\theta^m}{w^m}}{\frac{w_\theta^s}{w^s}} > 0$ or

$$\frac{w_\theta^s}{w^s} - \eta \cdot \frac{w_\theta^m}{w^m} = \frac{w_\theta^m}{w^m} \cdot (1 - \eta) - \int_{q^s}^{q^m} (\frac{w_{q\theta}}{w} - \frac{w_q \cdot w_\theta}{w^2}) \cdot dq > 0$$

Thus, given η, a sufficient condition for $\frac{dq^s}{d\theta} < 0$ (or $\frac{dq^s}{d\theta} > 0$) is that $w_{q\theta} < w_{q\theta}^{++}$ (or $w_{q\theta} > w_{q\theta}^{++}$), where

$$w_{q\theta}^{++} \equiv \frac{w_q \cdot w_\theta}{w} + \frac{w_\theta^m}{w^m} \cdot \frac{1 - \eta}{q^m - q^s} \cdot w > 0.$$

1.2.1 Single Crossing Property for opting-in households

Opting-in households do not privately demand any input, thus only the direct effect of θ on the marginal rate of substitution between net and gross income is relevant; hence

$$\frac{d\frac{dx}{dy}|_U}{d\theta} = \frac{\partial\frac{dx}{dy}|_U}{\partial\theta} < 0$$

as shown in Lemma 9.

1.3 Incentive Compatibility of First Best Allocations

Given individual optimization of private input investment, first best allocations can be characterized by the maximization of a social welfare function under government's budget constraint:

$$\max_{\underline{x},\underline{y},\overline{x},\overline{y}} \lambda \cdot V(\underline{x},\underline{y},0,\underline{\theta}) + (1-\lambda)\cdot V(\overline{x},\overline{y},0,\overline{\theta})$$
$$s.t. \quad \lambda \cdot (\underline{y}-\underline{x}) + (1-\lambda)\cdot(\underline{y}-\underline{x}) \geq 0 \quad (\mu)$$

by the first order conditions, the following optimization conditions arise

$$\frac{dx}{dy} \mid_V = 1$$

(and, by individual optimization, also $\frac{w}{y \cdot w_q} = 1$), for all $\theta \in \{\underline{\theta}, \overline{\theta}\}$.

Lemma 10 *The first best allocation is incentive-compatible for low-capability households.*

Proof. Let T be the first best income transfer received by low-capability households, thus by government budget constraint, $\underline{x} = \underline{y} + T$ and $\overline{x} = \overline{y} - \frac{\lambda}{1-\lambda} \cdot T$. The first best allocation is incentive compatible for low-capability individuals if

$$U(\underline{y}+T-q^m(\underline{y}+T,\underline{y},0,\underline{\theta}), \frac{y}{w(\underline{\theta}, q^m(\underline{y}+T,\underline{y},0,\underline{\theta}))}) \geq \tag{15}$$
$$U(\overline{y}-\frac{\lambda}{1-\lambda}\cdot T - q^m(\overline{y}-\frac{\lambda}{1-\lambda}\cdot T,\overline{y},0,\underline{\theta}), \frac{\overline{y}}{w(\underline{\theta}, q^m(\overline{y}-\frac{\lambda}{1-\lambda}\cdot T,\overline{y},0,\underline{\theta}))})$$

Given that $q^m(x,y,0,\underline{\theta})$ is the optimal quantity of input demanded by low-capability households under $\{x,y\}$ - hence, the first order condition (1) - is always satisfied - we apply the envelope theorem and write (15) as

$$\int_{\underline{y}}^{\overline{y}} \hat{U}_c(T)\cdot(1+\frac{\hat{U}_l(T)}{\hat{U}_c(T)\cdot\hat{w}(T)})\cdot dy - \int_{-\frac{\lambda}{1-\lambda}\cdot T}^{T} \hat{U}_c(\overline{y})\cdot d\tau \leq 0 \tag{16}$$

where

$$\hat{U}_c(T) \equiv \frac{\partial}{\partial c}U(y+T-q^m(y+T,y,0,\underline{\theta}),\frac{y}{w(\underline{\theta},q^m(y+T,y,0,\underline{\theta}))})$$
$$\hat{U}_l(T) \equiv \frac{\partial}{\partial l}U(y+T-q^m(y+T,y,0,\underline{\theta}),\frac{y}{w(\underline{\theta},q^m(y+T,y,0,\underline{\theta}))})$$
$$\hat{w}(T) \equiv w(\underline{\theta},q^m(y+T,y,0,\underline{\theta}))$$
$$\hat{U}_c(\overline{y}) \equiv \frac{\partial}{\partial c}U(\overline{y}+\tau-q^m(\overline{y}+\tau,y,0,\underline{\theta}),\frac{\overline{y}}{w(\underline{\theta},q^m(\overline{y}+\tau,\overline{y},0,\underline{\theta}))})$$

for all $y \in [\underline{y},\overline{y}]$ and all $\tau \in [-\frac{\lambda}{1-\lambda}\cdot T, T]$. By the SCP, if $\underline{y} < \overline{y}$ (or $\underline{y} > \overline{y}$) $1+\frac{\hat{U}_l(T)}{\hat{U}_c(T)\cdot\hat{w}(T)} < 0$ (or $1+\frac{\hat{U}_l(T)}{\hat{U}_c(T)\cdot\hat{w}(T)} > 0$), thus (16) is always satisfied with strict inequality. ■

As regards the first best tax schedule for households with high exogenous capability, we cannot conclude anything, given that - by the same kind of arguments used in Lemma 10:

$$U(\overline{y}-\frac{\lambda}{1-\lambda}\cdot T-q^m(\overline{y}-\frac{\lambda}{1-\lambda}\cdot T,\overline{y},0,\overline{\theta}),\frac{\overline{y}}{w(\overline{\theta},q^m(\overline{y}-\frac{\lambda}{1-\lambda}\cdot T,\overline{y},0,\overline{\theta}))})+$$
$$-U(\underline{y}+T-q^m(\underline{y}+T,\underline{y},0,\overline{\theta}),\frac{\underline{y}}{w(\overline{\theta},q^m(\underline{y}+T,\underline{y},0,\overline{\theta}))}) =$$
$$\underbrace{\int_{\underline{y}}^{\overline{y}}\hat{U}_c(-T)\cdot(1+\frac{\hat{U}_l(-T)}{\hat{U}_c(-T)\cdot\hat{w}(-T)})\cdot dy}_{>0}-\underbrace{\int_{-\frac{\lambda}{1-\lambda}\cdot T}^{T}\hat{U}_c(\overline{y})\cdot d\tau}_{>0}$$

In our analysis, we assume that the tax schedule for high-ability households is incentive incompatible. The same arguments hold also when we consider the public provision of private input. In the inclusive case, the SCP holds and all the above arguments apply, given that individuals do not control q any more. In the discriminating case, the low-ability households opting for the second-pillar provision receive at least the opt-out utility, thus the incentive constraint is satisfied. Also high-ability households opting for the second pillar provision receive at least their opting-out utility, which may reduce the incentive problem characterizing second best redistribution.

1.4 Optimal Tax and Public Provision

In the following, the optimization problems under different policy regimes are considered.

1.4.1 Pure Taxation Regime

The government's program is

$$\max_{\{\underline{x},\underline{y},\overline{x},\overline{y}\}} \lambda \cdot V(\underline{x},\underline{y},q^f,\underline{\theta}) + (1-\lambda)\cdot V(\overline{x},\overline{y},q^f,\overline{\theta})$$

$$s.t.$$

$$\lambda\cdot(\underline{y}-\underline{x}) + (1-\lambda)\cdot(\overline{y}-\overline{x}) \geq 0 \quad (\mu)$$

$$V(\overline{x},\overline{y},q^f,\overline{\theta}) \geq V(\underline{x},\underline{y},q^f,\overline{\theta}) \quad (\nu)$$

Thus, the Lagrangian is

$$\mathcal{L} = \lambda\cdot\underline{V} + (1-\lambda)\cdot\overline{V} + \mu\cdot[\lambda\cdot(\underline{y}-\underline{x}) + (1-\lambda)\cdot(\overline{y}-\overline{x})] + \nu\cdot(\overline{V}-\widehat{V})$$

where $\underline{V} \equiv V(\underline{x},\underline{y},q^f,\underline{\theta})$, $\overline{V} \equiv V(\overline{x},\overline{y},q^f,\overline{\theta})$, and $\widehat{V} \equiv V(\underline{x},\underline{y},q^f,\overline{\theta})$. The first order conditions are

$$\frac{\partial\mathcal{L}}{\partial\underline{x}} = \lambda\cdot\underline{V}_x - \lambda\cdot\mu - \nu\cdot\widehat{V}_x = 0 \tag{17}$$

$$\frac{\partial\mathcal{L}}{\partial\underline{y}} = \lambda\cdot\underline{V}_y + \lambda\cdot\mu - \nu\cdot\widehat{V}_y = 0 \tag{18}$$

$$\frac{\partial\mathcal{L}}{\partial\overline{x}} = (1-\lambda)\cdot\overline{V}_x - (1-\lambda)\cdot\mu + \nu\cdot\overline{V}_x = 0 \tag{19}$$

$$\frac{\partial\mathcal{L}}{\partial\overline{y}} = (1-\lambda)\cdot\overline{V}_y + (1-\lambda)\cdot\mu + \nu\cdot\overline{V}_y = 0 \tag{20}$$

By (19) and (20), $-\frac{\overline{V}_y}{\overline{V}_x} = \frac{dx}{dy}\mid_{\overline{V}} = 1$. By (17) and (18), after some algebra we obtain

$$\frac{dx}{dy}\mid_{\underline{V}} = 1 - \frac{\frac{\nu}{\lambda}\cdot\frac{\widehat{V}_x}{\underline{V}_x}}{1-\frac{\nu}{\lambda}\cdot\frac{\widehat{V}_x}{\underline{V}_x}}\cdot\left(\frac{dx}{dy}\mid_{\underline{V}} - \frac{dx}{dy}\mid_{\widehat{V}}\right)$$

where $\frac{dx}{dy}\mid_{\underline{V}}\equiv -\frac{\underline{V}_y}{\underline{V}_x}$ and $\frac{dx}{dy}\mid_{\widehat{V}}\equiv -\frac{\widehat{V}_y}{\widehat{V}_x}$. By (17), $1-\frac{\nu}{\lambda}\cdot\frac{\widehat{V}_x}{\underline{V}_x}>0$, and by the SCP, $\frac{dx}{dy}\mid_{\underline{V}} - \frac{dx}{dy}\mid_{\widehat{V}}>0$, hence $\frac{dx}{dy}\mid_{\underline{V}}<1$ (the optimal distortion implies a positive marginal tax on low-ability labor supply).

1.4.2 Inclusive Regime

Let $q=q^f+q^s$ be the total provision of public good. Under this regime the optimization problem is

$$
\begin{aligned}
\max_{\{\underline{x},\underline{y},\overline{x},\overline{y},q\}} \lambda\cdot U(\underline{x},\frac{\underline{y}}{w(\underline{\theta},q)})+(1-\lambda)\cdot U(\overline{x},\frac{\overline{y}}{w(\overline{\theta},q)}) \\
s.t. \\
\lambda\cdot(\underline{y}-\underline{x})+(1-\lambda)\cdot(\overline{y}-\overline{x})-q\geq 0 \quad (\mu) \\
U(\overline{x},\frac{\overline{y}}{w(\overline{\theta},q)})\geq U(\underline{x},\frac{\underline{y}}{w(\overline{\theta},q)}) \quad (\nu) \\
q\geq q_{\max} \quad (\eta)
\end{aligned}
$$

where $q_{\max}\equiv\{q^f+q^s\mid \max\{q^s(\underline{x},\underline{y},q^f,\underline{\theta}),q^s(\overline{x},\overline{y},q^f,\overline{\theta})\}\leq q^s\}$.

The Lagrangian is

$$
\mathcal{L}=\lambda\cdot\underline{U}+(1-\lambda)\cdot\overline{U}+\mu\cdot[\lambda\cdot(\underline{y}-\underline{x})+(1-\lambda)\cdot(\overline{y}-\overline{x})-q]+\nu\cdot(\overline{U}-\widehat{U})+\eta\cdot(q-q_{\max})
$$

where $\underline{U}\equiv U(\underline{x},\frac{\underline{y}}{\underline{w}})$, $\overline{U}\equiv U(\overline{x},\frac{\overline{y}}{\overline{w}})$, $\widehat{U}\equiv U(\underline{x},\frac{\underline{y}}{\overline{w}})$, $\underline{w}\equiv w(\underline{\theta},q)$, and $\overline{w}\equiv w(\overline{\theta},q)$. The first order conditions are

$$\frac{\partial \mathcal{L}}{\partial \underline{x}} = \lambda \cdot \underline{U}_c - \lambda \cdot \mu - \nu \cdot \widehat{U}_c = 0 \quad (21)$$

$$\frac{\partial \mathcal{L}}{\partial \underline{y}} = \lambda \cdot \frac{\underline{U}_l}{\underline{w}} + \lambda \cdot \mu - \nu \cdot \frac{\widehat{U}_l}{\overline{w}} = 0 \quad (22)$$

$$\frac{\partial \mathcal{L}}{\partial \overline{x}} = (1-\lambda+\nu) \cdot \overline{U}_c - (1-\lambda) \cdot \mu = 0 \quad (23)$$

$$\frac{\partial \mathcal{L}}{\partial \overline{y}} = (1-\lambda+\nu) \cdot \frac{\overline{U}_l}{\overline{w}} + (1-\lambda) \cdot \mu = 0 \quad (24)$$

$$\frac{\partial \mathcal{L}}{\partial q} = -\lambda \cdot \underline{U}_l \cdot \frac{y}{\underline{w}^2} \cdot \underline{w}_q - (1-\lambda+\nu) \cdot \overline{U}_l \cdot \frac{\overline{y}}{\overline{w}^2} \cdot \overline{w}_q - \mu + \nu \cdot \widehat{U}_l \cdot \frac{y}{\overline{w}^2} \cdot \overline{w}_q + \eta = 0. \quad (25)$$

By (23) and (24), $\frac{dx}{dy} \mid_{\overline{U}} \equiv -\frac{\overline{U}_l}{\overline{U}_x \cdot \overline{w}} = 1$. By (21) and (22), after some algebra we obtain that $\frac{dx}{dy} \mid_{\underline{U}} \equiv -\frac{\underline{U}_l}{\underline{U}_x \cdot \underline{w}} < 1$.

[INCOMPLETE HERE].

1.4.3 Discriminating Regimes

We consider first the case of low-ability in the second pillar and high-ability out. The optimization program of the government is

$$\max_{\{\underline{x},\underline{y},\overline{x},\overline{y},q^f,q^s\}} \lambda \cdot U(\underline{x}, \frac{\underline{y}}{w(\underline{\theta}, q^f + q^s)}) + (1-\lambda) \cdot V(\overline{x}, \overline{y}, q^f, \overline{\theta})$$

$$s.t.: \quad (26)$$

$$\lambda \cdot (\underline{y} - \underline{x}) + (1-\lambda) \cdot (\overline{y} - \overline{x}) - q^f - \lambda \cdot q^s \geq 0 \quad (\mu)$$

$$V(\overline{x}, \overline{y}, q^f, \overline{\theta}) \geq U(\underline{x}, \frac{\underline{y}}{w(\overline{\theta}, q^f + q^s)}) \quad (\nu)$$

$$q^s(\overline{x}, \overline{y}, q^f, \overline{\theta}) \geq q^s \quad (\overline{\eta})$$

$$q^s \geq q^s(\underline{x}, \underline{y}, q^f, \underline{\theta}) \quad (\underline{\eta})$$

$$q^f \geq 0 \quad (\underline{\varphi})$$

$$q^s(\overline{x}, \overline{y}, q^f, \overline{\theta}) \geq 0 \quad (\overline{\varphi})$$

hence, the corresponding Lagrangian is

$$\mathcal{L} = \lambda \cdot \underline{U} + (1-\lambda) \cdot \overline{V} + \mu \cdot [\lambda \cdot (\underline{y} - \underline{x}) + (1-\lambda) \cdot (\overline{y} - \overline{x}) - q^f - \lambda \cdot q^s] +$$
$$+\nu \cdot (\overline{V} - \widehat{U}) + \overline{\eta} \cdot (\overline{q}^s - q^s) + \underline{\eta} \cdot (q^s - \underline{q}^s) + \underline{\varphi} \cdot q^f + \overline{\varphi} \cdot \overline{q}^s$$

where $\underline{U} \equiv U(\underline{x}, \frac{y}{\underline{w}})$, $\overline{V} \equiv V(\overline{x}, \overline{y}, q^f, \overline{\theta})$, $\widehat{U} \equiv U(\underline{x}, \frac{y}{\widehat{w}})$, $\underline{w} \equiv w(\underline{\theta}, q^f + q^s)$, and $\widehat{w} \equiv w(\overline{\theta}, q^f + q^s)$. The first order conditions are

$$\frac{\partial \mathcal{L}}{\partial \underline{x}} = \lambda \cdot \underline{U}_c - \lambda \cdot \mu - \nu \cdot \widehat{U}_c - \underline{\eta} \cdot \underline{q}^s_x = 0 \tag{27}$$

$$\frac{\partial \mathcal{L}}{\partial \underline{y}} = \lambda \cdot \frac{\underline{U}_l}{\underline{w}} + \lambda \cdot \mu - \nu \cdot \frac{\widehat{U}_l}{\widehat{w}} - \underline{\eta} \cdot \underline{q}^s_y = 0 \tag{28}$$

$$\frac{\partial \mathcal{L}}{\partial \overline{x}} = (1-\lambda+\nu) \cdot \overline{V}_x - (1-\lambda) \cdot \mu + (\overline{\eta} + \overline{\varphi}) \cdot \overline{q}^s_x = 0 \tag{29}$$

$$\frac{\partial \mathcal{L}}{\partial \overline{y}} = (1-\lambda+\nu) \cdot \overline{V}_y + (1-\lambda) \cdot \mu + (\overline{\eta} + \overline{\varphi}) \cdot \overline{q}^s_y = 0 \tag{30}$$

$$\frac{\partial \mathcal{L}}{\partial q^f} = (1-\lambda+\nu) \cdot \overline{V}_{q^f} + \nu \cdot \widehat{U}_l \cdot \frac{y}{\widehat{w}^2} \cdot \widehat{w}_q - \lambda \cdot \underline{U}_l \cdot \frac{y}{\underline{w}^2} \cdot \underline{w}_q - \mu +$$
$$+(\overline{\eta} + \overline{\varphi}) \cdot \overline{q}^s_{q^f} - \underline{\eta} \cdot \underline{q}^s_{q^f} + \underline{\varphi} = 0 \tag{31}$$

$$\frac{\partial \mathcal{L}}{\partial q^s} = -\lambda \cdot \underline{U}_l \cdot \frac{y}{\underline{w}^2} \cdot \underline{w}_q - \mu \cdot \lambda + \nu \cdot \widehat{U}_l \cdot \frac{y}{\widehat{w}^2} \cdot \widehat{w}_q - \overline{\eta} + \underline{\eta} = 0 \tag{32}$$

By (29) and (30), we get

$$\frac{dx}{dy} \mid_{\overline{V}} = 1 + \frac{\overline{\eta} + \overline{\varphi}}{1-\lambda+\nu} \cdot \frac{\overline{q}^s_x + \overline{q}^s_y}{\overline{V}_x} \tag{33}$$

where $\frac{dx}{dy} \mid_{\overline{V}} \equiv -\frac{\overline{V}_y}{\overline{V}_x}$. Also, by (27) and (28), we get

$$\frac{dx}{dy} \mid_{\underline{U}} = 1 - \frac{\frac{\nu}{\lambda} \cdot \frac{\widehat{U}_c}{\underline{U}_c} \cdot \left(\frac{dx}{dy} \mid_{\underline{U}} - \frac{dx}{dy} \mid_{\widehat{U}} \right) + \frac{\underline{\eta}}{\lambda \cdot \underline{U}_c} \cdot (\underline{q}^s_x + \underline{q}^s_y)}{1 - \frac{\nu}{\lambda} \cdot \frac{\widehat{U}_c}{\underline{U}_c}} \tag{34}$$

where $\frac{dx}{dy} \mid_{\underline{U}} \equiv -\frac{\underline{U}_l}{\underline{U}_c \cdot \underline{w}}$ and $\frac{dx}{dy} \mid_{\widehat{U}} \equiv -\frac{\widehat{U}_l}{\widehat{U}_c \cdot \widehat{w}}$.

We now prove the Propositions of Sections 4.1.1.

Proof of Proposition 1. As regards the first part of the Proposition, substituting (27) and $\overline{\eta} = \underline{\eta} = 0$ in (32), after some algebra we obtain $\underline{\phi} = \frac{\frac{\nu}{\lambda}}{1-\frac{\nu}{\lambda}} \cdot (\widehat{\phi} - \underline{\phi})$, where $\underline{\phi} = -\underline{U}_c - \underline{U}_l \cdot \frac{y}{\underline{w}^2} \cdot \underline{w}_q$ and $\widehat{\phi} = -\widehat{U}_c - \widehat{U}_l \cdot \frac{\widehat{y}}{\widehat{w}^2} \cdot \widehat{w}_q$. By comparative statics on household's optimization problem (Lemma 8), we know that $\phi_\theta > 0$ - hence, $\widehat{\phi} - \underline{\phi} > 0$ and - (respectively, $\phi_\theta < 0$ and $\widehat{\phi} - \underline{\phi} < 0$) when capability and input are complements (respectively, substitutes). By (27) and (28), $\nu = \frac{\lambda \cdot (\underline{U}_c + \frac{\underline{U}_l}{\underline{w}}) - \underline{\eta} \cdot (\underline{q}^s_x + \underline{q}^s_y)}{\widehat{U}_c + \frac{\widehat{U}_l}{\widehat{w}}}$. Given that private consumption is a normal good, $\underline{U}_c + \frac{\underline{U}_l}{\underline{w}} < \widehat{U}_c + \frac{\widehat{U}_l}{\widehat{w}}$ and $\nu < \lambda$. Therefore, the (public) provision of input is below (resp., above) the optimal level that low-capability households would choose on their own when capability and input are complements, hence $\underline{\phi} > 0$ (resp., substitutes, hence $\underline{\phi} < 0$).

Assume that an optimal public provision mix exists such that $q^{f*} > 0$ and $q^{s*} \geq 0$. Consider now a marginal perturbation of the optimal public provision mix such that: the first pillar provision is decreased by $dq^f < 0$, the second-pillar provision is increased by $dq^s = -dq^f > 0$, and the net-of-tax income of high-capability households is increased by $d\overline{x} = -dq^f > 0$. It is easy to check that the marginal perturbation leaves unaffected the total amount of publicly provided input - hence the welfare of low-capability households, the welfare and optimal behavior of high-capability households, and the government budget balance. Therefore, the social welfare associated to new optimal public provision mix (where the first-pillar provision is lower and the second-pillar provision is higher) is the same as under the initial optimal provision (and taxation) policy.

The second result of the Proposition is easily obtained by inspection of conditions (33) and (34), contrasted respectively with conditions obtained by the PT optimal taxation problem (17), (18), (19), and (20). ■

Let us now consider the case of corner solutions, where the scope for the second-pillar provision is constrained by lower and upper bounds on q^f and q^s.

Lemma 11 *$\overline{\eta} > 0$ (i.e., $q^{s*} = \overline{q}^s$) if and only if $\overline{\varphi} > 0$ (i.e., $\overline{q}^s = 0$). Moreover, $\underline{\eta} > 0$ (i.e., $q^{s*} = \underline{q}^s$) if and only if $\underline{\varphi} > 0$ (i.e., $q^{f*} = 0$).*

Proof. Substituting (29) and (31) in (32), observing that $\overline{V}_x = \overline{V}_{qf}$, and by definition of $\overline{q}^s_x$ and $\overline{q}^s_{qf}$, we get

$$\overline{\eta} \cdot \underbrace{(1 - \overline{q}^s_x + \overline{q}^s_{qf})}_{>0} + \underline{\varphi} = \underline{\eta} \cdot \underbrace{(1 + \underline{q}^s_{qf})}_{>0} + \overline{\varphi} \cdot \underbrace{(\overline{q}^s_x - \overline{q}^s_{qf})}_{>0} \geq 0. \tag{35}$$

■

Proof of Proposition 2. By Lemma 11, we know that only two types of corner solutions are possible. Assume that $\underline{\eta} > 0$ and $\underline{\varphi} > 0$ when capability and input are economic substitutes. The considered corner solution is such that: $q^{f*} > 0$ and $q^{s*} = \underline{q}^s > 0$; thus, $\underline{\phi} > 0$. By economic substitutability between capability and input, $\widehat{\phi} - \underline{\phi} < 0$ (see the argument of Proposition 1). Thus, we have a contradiction: $0 \leq \underline{\phi} < \frac{\frac{\nu}{\lambda}}{1-\frac{\nu}{\lambda}} \cdot (\widehat{\phi} - \underline{\phi}) < 0$. The only corner solution compatible with economic substitutability between capability and input is $\overline{\eta} > 0$ and $\overline{\varphi} > 0$, hence $\overline{q}^s = 0$. Remark that this implies also $\overline{q}^m = 0$. Moreover, by inspection of (33) and (34), also the second result is proven. ■

Proof of Proposition 3. The argument is similar to Proposition 2. Assume that the corner solution featured by $\overline{\eta} > 0$ and $\overline{\varphi} > 0$ is compatible with economic complementarity between capability and input. Then, the optimal public provision mix is pure topping up, and such that $\overline{q}^s = \overline{q}^m = 0$ (hence, $\overline{\phi} = 0$). By economic complementarity between capability and input, q^{f*} is such that $\underline{\phi} < 0$. Thus, we have a contradiction, given that $\overline{\eta} > 0$ and economic complementarity, necessarily $\underline{\phi} > \frac{\frac{\nu}{\lambda}}{1-\frac{\nu}{\lambda}} \cdot (\widehat{\phi} - \underline{\phi}) > 0$. Thus, the only corner solution compatible with economic complementarity between capability and input is $\underline{\eta} > 0$ and $\underline{\varphi} > 0$, that imply $q^{s*} = \underline{q}^s > 0$ and $q^{f*} = 0$. The second result derives by inspection of (33) and (34). ■

We now consider the DMH regime. In this case, government's program is

$$\max_{\{\underline{x},\underline{y},\overline{x},\overline{y},q^f,q^s\}} \lambda \cdot V(\underline{x},\underline{y},q^f,\underline{\theta}) + (1-\lambda)\cdot U(\overline{x},\frac{\overline{y}}{w(\overline{\theta},q^f+q^s)})$$
$$s.t.: \tag{36}$$
$$\lambda\cdot(\underline{y}-\underline{x}) + (1-\lambda)\cdot(\overline{y}-\overline{x}) - q^f - (1-\lambda)\cdot q^s \geq 0 \quad (\mu)$$
$$U(\overline{x},\frac{\overline{y}}{w(\overline{\theta},q^f+q^s)}) \geq V(\underline{x},\underline{y},q^f,\overline{\theta}) \quad (\nu)$$
$$q^s(\underline{x},\underline{y},q^f,\underline{\theta}) \geq q^s \quad (\underline{\eta})$$
$$q^s \geq q^s(\overline{x},\overline{y},q^f,\overline{\theta}) \quad (\overline{\eta})$$
$$q^f \geq 0 \quad (\underline{\varphi})$$
$$q^s(\underline{x},\underline{y},q^f,\underline{\theta}) \geq 0 \quad (\overline{\varphi})$$

The Lagrangian of this program is

$$\mathcal{L} = \lambda\cdot\underline{V} + (1-\lambda)\cdot\overline{U} + \mu\cdot[\lambda\cdot(\underline{y}-\underline{x}) + (1-\lambda)\cdot(\overline{y}-\overline{x}) - q^f - (1-\lambda)\cdot q^s] +$$
$$+\nu\cdot(\overline{U}-\widehat{V}) + \underline{\eta}\cdot(\underline{q}^s - q^s) + \overline{\eta}\cdot(q^s - \overline{q}^s) + \underline{\varphi}\cdot q^f + \overline{\varphi}\cdot\underline{q}^s$$

where $\underline{V} \equiv V(\underline{x},\underline{y},q^f,\underline{\theta})$, $\overline{U} \equiv U(\overline{x},\frac{\overline{y}}{\overline{w}})$, $\widehat{V} \equiv V(\underline{x},\underline{y},q^f,\overline{\theta})$, and $\overline{w} = w(\overline{\theta},q^f+q^s)$. The first order conditions are

$$\frac{\partial\mathcal{L}}{\partial\underline{x}} = \lambda\cdot\underline{V}_x - \lambda\cdot\mu - \nu\cdot\widehat{V}_x - \underline{\eta}\cdot\underline{q}^s_x + \overline{\varphi}\cdot\underline{q}^s_x = 0 \tag{37}$$
$$\frac{\partial\mathcal{L}}{\partial\underline{y}} = \lambda\cdot\underline{V}_y + \lambda\cdot\mu - \nu\cdot\widehat{V}_y - \underline{\eta}\cdot\underline{q}^s_y + \overline{\varphi}\cdot\underline{q}^s_y = 0 \tag{38}$$
$$\frac{\partial\mathcal{L}}{\partial\overline{x}} = (1-\lambda+\nu)\cdot\overline{U}_c - (1-\lambda)\cdot\mu - \overline{\eta}\cdot\overline{q}^s_x = 0 \tag{39}$$
$$\frac{\partial\mathcal{L}}{\partial\overline{y}} = (1-\lambda+\nu)\cdot\frac{\overline{U}_l}{\overline{w}} + (1-\lambda)\cdot\mu - \overline{\eta}\cdot\overline{q}^s_y = 0 \tag{40}$$
$$\frac{\partial\mathcal{L}}{\partial q^f} = \lambda\cdot\underline{V}_{q^f} - (1-\lambda+\nu)\cdot\overline{U}_l\cdot\frac{\overline{y}}{\overline{w}^2}\cdot\overline{w}_q - \mu - \nu\cdot\widehat{V}_{q^f} + (\underline{\eta}+\overline{\varphi})\cdot\underline{q}^s_{q^f} - \overline{\eta}\cdot\overline{q}^s_{q^f} + \underline{\varphi} = 0 \tag{41}$$
$$\frac{\partial\mathcal{L}}{\partial q^s} = -(1-\lambda+\nu)\cdot\overline{U}_l\cdot\frac{\overline{y}}{\overline{w}^2}\cdot\overline{w}_q - \mu\cdot(1-\lambda) - \underline{\eta} + \overline{\eta} = 0. \tag{42}$$

By (39) and (40),

$$\frac{dx}{dy} \mid \overline{U} = -\frac{\overline{U}_l}{\overline{U}_c \cdot \overline{w}} = 1 - \overline{\eta} \cdot \frac{\overline{q}_x^s + \overline{q}_x^s}{(1-\lambda+\nu) \cdot \overline{U}_c}$$

And, by (37) and (38),

$$\frac{dx}{dy} \mid \underline{V} = -\frac{\underline{V}_y}{\underline{V}_x} = 1 - \frac{\frac{\nu}{\lambda} \cdot \frac{\widehat{V}_x}{\underline{V}_x} \cdot \left(\frac{dx}{dy} \mid_{\underline{V}} - \frac{dx}{dy} \mid_{\widehat{V}}\right) + \frac{\eta+\overline{\varphi}}{\lambda \cdot \underline{V}_x} \cdot (\underline{q}_x^s + \underline{q}_y^s)}{1 - \frac{\nu}{\lambda} \cdot \frac{\widehat{V}_x}{\underline{V}_x}}$$

Proof of Proposition 4. At the optimum, $\overline{\eta} = 0$. Assume conversely that $\overline{\eta} > 0$ (hence $\underline{\eta} = 0$), then necessarily $q^s = \overline{q}^s$, hence $-\overline{U}_c - \overline{U}_l \cdot \frac{\overline{y}}{\overline{w}^2} \cdot \overline{w}_q > 0$, by construction of $\overline{q}^s$. By (39) and (42),

$$-\overline{U}_c - \overline{U}_l \cdot \frac{\overline{y}}{\overline{w}^2} \cdot \overline{w}_q = \frac{\underline{\eta} - \overline{\eta} \cdot (1 + \underline{q}_x^s)}{1 - \lambda + \nu}$$

that implies a contradiction. Therefore, the marginal tax rate on high-capability households is zero. Moreover, at the optimum, also $\underline{\eta} = 0$. Assume by contradiction that $\underline{\eta} > 0$, then necessarily $q^s = \underline{q}^s$. This would imply that $-\overline{U}_c - \overline{U}_l \cdot \frac{\overline{y}}{\overline{w}^2} \cdot \overline{w}_q > 0$. Let q^f and q^s be the optimal first- and second-pillar provision at the optimum such that $\underline{\eta} > 0$. It is easy to see that such a policy mix cannot be optimal. Consider the following marginal policy reform: the first-pillar public provision increases by $dq^f > 0$; the tax on low-capability households is increased lump-sum accordingly, $\frac{d\underline{x}}{dq^f} = 1$; and the tax on high-capability households is increased (lump sum) by $\frac{d\overline{x}}{dq^f} = 1 - \frac{d\overline{q}^s}{dq^f}$. Remark also that publicly provided input (to high-capability households) is necessarily reduced by $\frac{d\overline{q}^s}{dq^f}$ to prevent low-capability households from opting in. Remark also that the considered policy reforms keeps the equilibrium of government's budget. Thus, the total welfare increases

$$\lambda \cdot \underbrace{(\underline{V}_x \cdot \frac{d\underline{x}}{dq^f} + \underline{V}_{q^f})}_{=0} + (1-\lambda) \cdot \underbrace{(-\overline{U}_c - \overline{U}_l \cdot \frac{\overline{y}}{\overline{w}^2} \cdot \overline{w}_q)}_{>0} \cdot (1 - \frac{d\overline{q}^s}{dq^f})$$

Hence, we have a contradiction. The optimal policy mix necessarily requires $-\overline{U}_c - \overline{U}_l \cdot$

$\frac{\overline{y}}{\overline{w}^2} \cdot \overline{w}_q = 0$, then by (37), (39) and (41) we also see that

$$(\underline{\eta} + \overline{\varphi}) \cdot (\underline{q}^s_x - \underline{q}^s_{q^f}) + \overline{\eta} \cdot \overline{q}^s_{q^f} - \underline{\varphi} = 0 \tag{43}$$

hence, by $\underline{\eta} = \overline{\eta} = 0$ necessarily $\underline{\varphi} = \overline{\varphi} = 0$. Therefore, the marginal tax wedge on low-capability households is as in the PT regime. Remark that the optimal public provision mix is equivalent to what high-capability households would do on their own in absence of publicly provided input. ■

1.4.4 Multi-class Economy

Let

$$\hat{\theta} \equiv \{\theta \mid U(x(\theta), \frac{y(\theta)}{w(\theta, q^f + q^s)}) = V(x(\theta), y(\theta), q^f, \theta)\}.$$

Remark that, given the (optimal) tax schedule for a given type θ, $\frac{dx}{dy} \mid_\theta = 1 - \frac{\partial T(y(\theta))}{\partial y(\theta)}$, where $\frac{\partial T(y(\theta))}{\partial y(\theta)}$ is the marginal income tax. By the definition of $\hat{\theta}$:

$$\frac{d\hat{\theta}}{dx} = -\frac{U^s_c - U^m_c}{\Delta}$$
$$\frac{d\hat{\theta}}{dy} = -\frac{\frac{U^s_l}{w^s} - \frac{U^m_l}{w^m}}{\Delta}$$
$$\frac{d\hat{\theta}}{dq^f} = \frac{\frac{U^s_l}{w^{s2}} \cdot w^s_q - \frac{U^m_l}{w^{m2}} \cdot w^m_q}{\Delta} \quad \frac{d\hat{\theta}}{dq^s} = \frac{\frac{U^s_l}{w^{s2}} \cdot w^s_q}{\Delta}$$

where $\Delta \equiv \frac{-U^s_l}{w^{s2}} \cdot w^s_\theta + \frac{U^m_l}{w^{m2}} \cdot w^m_\theta$ is positive (or negative) only if $w_{q\theta} \leq w^{++}_{q\theta}$ (or $w_{q\theta} > w^{++}_{q\theta}$). Thus, when complementarity between q and θ is not too strong (or is strong enough):

- $\frac{d\hat{\theta}}{dx} > 0$ (or $\frac{d\hat{\theta}}{dx} < 0$)

- $\frac{d\hat{\theta}}{dy} < 0$ (or $\frac{d\hat{\theta}}{dy} > 0$)

- $\frac{d\hat{\theta}}{dq^f} < 0$ (or $\frac{d\hat{\theta}}{dq^f} > 0$)

- $\frac{d\hat{\theta}}{dq^s} < 0$ (or $\frac{d\hat{\theta}}{dq^s} > 0$).

By

$$x(y(\theta), u(\theta), q^f, q^s, \theta \equiv \{u(\theta) = U(x(\theta), \frac{y(\theta)}{w(\theta, q^f + q^s)})\}$$

if follows that

$$\frac{dx}{dy} = -\frac{U_l^s}{w^s \cdot U_c^s}$$
$$\frac{dx}{du} = \frac{1}{U_c^s}$$
$$\frac{dx}{dq^f} = \frac{dx}{dq^s} = -\frac{dx}{dy} \cdot \frac{w_q^s}{w^s}$$

By

$$x(y(\theta), v(\theta), q^f, \theta \equiv \{v(\theta) = V(x(\theta), y(\theta), q^f, \theta)\}$$

if follows that

$$\frac{dx}{dy} = -\frac{U_l^m}{w^m \cdot U_c^m}$$
$$\frac{dx}{du} = \frac{1}{U_c^m}$$
$$\frac{dx}{dq^f} = -\frac{dx}{dy} \cdot \frac{w_q^m}{w^m}.$$