THE DISTRIBUTIONAL INCIDENCE OF GROWTH: A NON-ANONYMOUS AND RANK DEPENDENT APPROACH

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The distributional incidence of growth: a non-anonymous and rank dependent approach*

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Abstract

This paper provides a normative framework for the assessment of the distributional incidence of growth. By removing the anonymity axiom, such framework is able to evaluate the individual income changes over time and the reshuffling of individuals along the income distribution that are determined by the pattern of income growth. We adopt a rank dependent social welfare function expressed in terms of initial individual rank and individual income changes and we obtain dominance conditions over different growth paths. Then we show that these conditions can be interpreted in terms of non-anonymous Growth Incidence Curve. Finally, we propose two families of indices measuring respectively the horizontal inequality of growth and its progressivity.

Keywords: Growth; Pro-poor; Inequality; Income mobility; Dominance

1 Introduction

In recent years a new literature has emerged, both theoretical and empirical, on the measurement of the distributive impact of growth (see Bourguignon, 2003, 2004; Ferreira, 2010; Son, 2004; Ravallion and Chen, 2003). Different tools (both scalar measures and dominance conditions) have been proposed for the evaluation of the pro-poorness of growth (see Duclos, 2009; Grosse et al., 2008; Kakwani and Son, 2008; Klasen and Misselhorn, 2008; Kraay, 2006; Son, 2004; Kakwani and Pernia, 2000; Essama-Nssah and Lambert, 2009; Zheng, 2010), and different decompositions have been obtained that relate the concepts of growth to poverty and inequality (see Datt and Ravallion; 1992; Jenkins and Van Kerm, 2006). One basic tool used in this literature is the growth incidence curve (GIC). GIC measures the quantile-specific rate of economic growth between two points in time as a function of each percentile (Ravallion and Chen,

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2003). A downward sloping GIC indicates that growth contributes to equalizing the distribution of income, whereas an upward sloping GIC indicates a non-equalizing growth. Ravallion and Chen (2003) and Son (2004) also provide first and second order dominance conditions for ranking growth processes according to the shape of their GIC.

In this literature the growth process is basically analyzed by comparing the pre-growth and post-growth distribution. There is a clear analogy here between the transformation of an income distribution throughout the effect of growth and the transformation obtained as the effect of an income tax. In the same way as progressive taxation reduces inequality, a "progressive" growth reduces the inequality in a distribution. In fact, such analogy has been deeply explored in the literature, and different well established results in the progressivity literature have been applied to this context (see Bénabou and Ok, 2001; Jenkins and Van Kerm, 2006, 2004; 2011; Van Kerm, 2006).

Now, with very few recent exceptions that will be discussed below, all this literature, when analyzing the distributional impact of growth, basically compares the phenomena under scrutiny before and after the growth process has taken place. Hence, for instance, to measure the pro-poorness of growth, the poverty levels (measured, say, according to the headcount or the poverty gap indices) are computed in the two periods of time and then compared. The same for different distributional indices. If this procedure is all right when one is interested in measuring the pure distributional change that takes place in a distribution, this is unsatisfactory if one is interested in evaluating growth in terms of social welfare: from this view point, it can make a big difference if the poor people in the first period are still the poor people in the following period, thereby experiencing chronic poverty, or if there has been a substantial reshuffling of the individual positions in the population. Now, to capture this aspect of the distributional impact of growth one needs to remove one basic axiom which is used in the literature: that is the axiom of anonymity. According to this axiom, poverty and inequality measures are required to be invariant to permutations of income vectors. As a consequence, the individual income dynamics along the distribution is ignored.

We believe, on the contrary, that for the welfare evaluation of the distributional impact of growth the identities of individuals do matter. Considering the identities of individuals allows us to study the individual income dynamics and the mobility that takes place during the growth process; aspects, both crucial for a welfare assessment of growth, which are hidden by the anonymity assumption.

This consideration is at center stage of our study.

In particular, we provide an analytical framework for a welfare analysis of income growth, within which individual income changes over time are related to the pattern of income growth and the reshuffling of individuals along the distribution. We argue that an income transformation process must be evaluated with respect to the initial economic situation of individuals. Therefore, the methodology we propose for evaluating the dynamic of income distributions, is based on some normative principles, which take into account the joint effect of the income transformation process and the identity of individuals, given by their initial economic condition. We provide partial rankings of social states and show that these conditions can be interpreted in terms of non-anonymous Growth Incidence Curve. Finally, we propose two families of indices measuring respectively the horizontal inequality of growth and its progressivity.

Three recent contributions are very closed in spirit to the approach we explore in this paper.

Grimm (2007) introduces the non-anonymous GIC (na-GIC) in the pro-poor growth literature[1], in order to investigate the relationship between growth and mobility. The na-GIC measures the individual-specific rate of economic growth between two points in time, thus comparing the income of individuals which were in the same initial position, independently of the position they acquire in the final distribution of income. The na-GIC is obtained by keeping the ranking of statistical units constant; whereas comparing the initial and terminal quantile functions, as the standard GIC does, is equivalent to reranking individuals, with the result that it is not the income of the same individual that is compared, but the income of the same quantile. Grimm (2007) computes standard GIC with non-anonymous GIC, showing how different conclusions emerge by adopting one instead of the other, but he does not propose any normative evaluation of growth.

Bourguignon (2010) applies a general framework, developed in a previous literature for the welfare evaluation of bi-dimensional distributions, to the analysis of growth. A growth process is represented by a bivariate distribution of initial and final income. He shows that the distributive effect of growth can be decomposed into two terms: differences between growth processes and differences between initial distributions. Then, by assuming the same base-initial distribution of income, he compares two growth processes and proposes a social welfare justification for the use of na-GIC, in the joint evaluation of growth and mobility. In his framework, both dimensions are aggregate according to a utilitarian social welfare function (SWF); a concern for inequality of income growth at different (initial) income levels is introduced by imposing different conditions on the marginal utility with respect to initial income and income change. Three kinds of dominance conditions are derived. One is a first order stochastic dominance condition of distributions of the income change of individuals belonging to the same initial income group, to be checked for every initial group. This condition is then shown to imply na-GIC dominance conditions. The second is a first order stochastic dominance of distributions of income change to be checked sequentially, by cumulating the initial income groups. The third is a second order stochastic dominance of distributions of income change, to be checked sequentially by cumulating all initial income groups, or what he calls p-cumulative income change curve dominance. These last two conditions are, then, shown to imply the cumulated na-GIC dominance. Lastly, he proposes to correct the mean income change curve, by an index capturing the horizon-

[1] Also the analysis proposed by Jenkins and Van Kerm (2006) and Van Kerm (2009) are based on this intuition.

tal inequity of growth. Therefore, while na-GIC represents a tool that can be implemented in order to introduce mobility in the standard growth evaluations framework, inequality corrected na-GIC, represent a tool that can be employed to express a concern for the inequality of growth.

Jenkins and Van Kerm (2011) propose dominance conditions to rank bivariate distributions of initial incomes and income changes (either relative or absolute). They propose a social welfare justification, based on a rank dependent SWF, for the adoption of mobility profiles, or na-GIC according to our terminology. The mobility profile is meant to be a plot of a generic measure of individual income change against the initial rank p; thus, mobility profiles reveal how income growth is distributed according to the position of individuals in the base year income distribution. Two dominance conditions are characterized: the first one is a first order dominance between two different mobility profiles; the second is a cumulated mobility profile dominance. Finally, they propose to measure the progressivity of growth as the difference between their rank dependent measure of growth and the arithmetic mean of the individuals' income growth, representing the income growth each individual would experience in the absence of an equalizing process.

In this paper we follow the intuition of the previous papers in assuming that a more comprehensive evaluation of growth should relax anonymity. We consider bivariate distributions of initial incomes and income changes; however, instead of adopting the utilitarian approach, we use a rank dependent approach to social welfare and inequality (see Yaari 1988). In order to leave out anonymity, we first classify the individuals according to the quantile group they belong to in the initial distribution of income; then, we evaluate the income evolution of individuals belonging to the same initial quantile group. New dominance criteria are obtained for the social welfare evaluation of growth. Some necessary conditions are derived that compare growth paths on the basis of different definitions of na-GIC. Furthermore, we propose to evaluate the horizontal inequality of growth and its progressivity by means of a family of generalized Gini indices.

The work is structured as follows. In section 2 we outline the framework. In section 3 we present the results. In section 4 we conclude.

2 The framework

We consider an initial distribution of income at time t, $\mathbf{y}$, with fixed population size. We denote by $F(y)$ the cumulative distribution function of $\mathbf{y}$ and by $f(y)$ its density. Growth takes place over some time period, such that, at time $t+1$, the final distribution of income is represented by $F(w)$. Our aim is to analyze the non-anonymous distributional impact of growth, which determines the transformation from $F(y)$ to $F(w)$ and requires to account for initial and terminal income. This extension implies that we need to consider the full joint distribution of individual initial and final income, or income change. In order for this to be done, we need to explain the concepts of growth path and growth process.

A growth path is described by a joint density function of initial income and final income, $f(y,w)$. Since $f(y,w) = f(y) f(w \mid y)$, a growth path can be defined as a combination of a transitional or conditional density function with an initial distribution. A growth process, instead, can be summarized by the transitional density function, $f(w \mid y)$. Therefore a growth path is a combination of a growth process with the initial distribution of income.

When the aim is to compare growth paths Bourguignon (2010) suggests to look at the following decomposition:

$$f(y,w) - f^*(y,w) = f(y)\,\Delta f(w \mid y) + \Delta f(y)\,f^*(w \mid y) \quad (1)$$

Eq. (1) states that the difference in "growth paths", $\Delta f(y,w) = f(y,w) - f^*(y,w)$, is the combination of growth processes, $\Delta f(w \mid y)$, with an initial distribution. The first part of (1) is the contribution to the difference in growth paths of the "growth processes", i.e. densities of terminal income conditional on initial income. The second part is the contribution of the difference in initial distributions for a given growth process.

Our aim is to rank growth paths, when the identity of individuals do matter.

We assume that we can order individuals according to their identity. This identity can be defined by their membership to a group, such a group being, for example, the quantile group they belong to in the initial distribution of income[2]. To this aim, we partition the population into groups of individuals homogeneous with respect to the quantile group they belong to in $F(y)$. We index each subgroup by $i = 1, ..., m$ in increasing order; hence q_i is the portion of individuals in quantile group i[3] and y_i is the income of individuals belonging to the quantile group i in $F(y)$, therefore i can also reveal information about individuals' rank in $F(y)$. For analytical convenience, instead of referring to $F(w \mid y)$, we shall refer to the distributional impact described by the distribution of income change, $x = w - y$, conditional on initial relative position of individuals before growth; That is, given a quantile group, we can observe the income change of each of its members and we can define a distribution of these individual income changes, which by construction is specific for each quantile group. For brevity we refer to this distribution as within group distribution of income change. Thus, let $\mathbf{x}_i$ the distribution of the income changes of individuals belonging to quantile group i in $F(y)$, and let $F_i(x)$ be its cumulative distribution function, we can express each x in terms of quantile function: $p = F_i(x) \Longleftrightarrow x_i(p) = F_i^{-1}(p)$, $F_i^{-1}(p) := \inf\{x : F_i(x) \geq p\}$ is the left inverse function, denoting the level of income change of individuals whose rank in the specific distribution of income change of individuals belonging to the same initial quantile group, i, is p.

Let D be the set of admissible growth paths.

We are interested in ranking members of D and we assume that such ranking can be represented by a social welfare function. However, given the properties

[2] Such as the poorest 5th.

[3] Since, by definition, quantiles partition the population equally, in what follows we will ignore this information.

of the social welfare function that we adopt and which will be explained below, the differences in the initial distribution of income are neutralized. This enables to evaluate growth paths that take place over different initial distributions of income, whitout growth being affected by the specific level of initial income.

Given the particular role played in our framework not only by the final but also by the initial economic situation of individuals, which can be expressed by their membership to an initial quantile group, we propose to aggregate both dimensions according to a rank-dependent social welfare function (Yaari, 1988). According to this formulation, social preferences over distributions of income changes of a specific initial quantile group are represented by a weighted average of ordered income changes, where each income change is weighted according to its position in the ranking. For a generic quantile group, this social welfare can be defined by[4]:

$$W\left(F_i\left(x\right)\right)=\int_0^1 v_i\left(p\right)F_i^{-1}\left(p\right)dp$$

The function $v_i\left(p\right):[0,1]\longrightarrow\Re_+$ expresses the weight attached by the society to any income change ranked at p within the distribution of initial quantile group i. $v_i\left(p\right)F_i^{-1}\left(p\right)$ is the contribution to social welfare of the fraction dp of individuals characterized by the same initial economic condition.

An overall social welfare evaluation of growth paths is obtained by applying an additional rank dependent aggregation procedure. That is, we aggregate the social welfare evaluation of growth experienced by each initial quantile group (see Zoli, 2000 and Peragine, 2002), weighted by the relevant population share, using quantile-specific weighting functions. We obtain the following YSWF expressing concern for both the initial and final economic situation of the individuals it represents:

$$W\left(F\left(x\mid y\right)\right)=\sum_{i=1}^{m}q_i\int_0^1 v_i\left(p\right)F_i^{-1}\left(p\right)dp \tag{2}$$

The income change structure of non-anonymous individuals is modelled by restriction on the weight function $v_i\left(p\right)$, which now depends on two information: the relative position in the initial distribution of income, as expressed by the subscript i, and the relative position, p, in the distribution of income change among the individuals belonging to the same initial quantile group[5]. The set of weight functions specifying the social welfare function, $< v_1(p), ..., v_n(p) >$, will

[4] See also Donaldson and Weymark (1980), Weymark (1981), Ebert (1987), Aaberge (2001).

[5] Note the main difference with Borguignon (2010). In fact, while he conditions income changes to initial income levels and requires information about the density distribution of the base-year initial income distribution, we do not require such information, since we condition income changes on initial rank. This implies that we can use our framework to compare growth paths that take place over different initial distributions of income.

be called a weight profile. Different value judgments are expressed in this framework by selecting different classes of "social weight" functions. These implicitly define welfare rankings.

2.1 Properties

In this section we discuss the properties that a YSWF must satisfy. Consequently, different families of YSWF are derived, depending of the restrictions imposed on the weight function.

The first property is a standard monotonicity assumption.

Axiom 1 (Pro-growth) *For all* $i = 1, ..., m$, *for all* $p \in [0, 1]$

$$v_i(p) \geq 0$$

This axiom states that social welfare does not decrease if an individual experiences an increment of income change (thus, for a given level of initial income, an increment of final income).

Note that since we work on distributions of income change, we can have situations where individuals experience negative income growth. The positivity of the weight grants neutrality with respect to the sign of the growth. Therefore, if an individual experiences positive income change, this income change multiplied by a positive weight generates an increment of social welfare. By contrast, if an individual experiences negative income growth, this income loss multiplied by a positive weight gives a reduction of social welfare, which is in line with the principle underlaying the pro-growth axiom.

Let $\mathbf{V_1} = \{< v_1(p), ..., v_m(p) >$: Axiom 1 holds$\}$ and let $\mathbf{W}_1$ be the class of YSWFs constructed as in (2) and based on weights profiles in $\mathbf{V}_1$.

We proceed with the axiom imposing an aversion to inequality in the initial income dimension.

Axiom 2 (Pro-poorness of growth) *For all* $p \in [0, 1]$, for all $i = 1, ..., m-1$

$$v_i(p) \geq v_{i+1}(p)$$

This is expression of the Pigou-Dalton principle of transfer among individuals having different membership in $F(y)$. That is, a transfer of a small amount of income change from an individual ranked p in a richer initial quantile group $i + 1$ to an individual ranked p in a poorer initial quantile group i does not decrease social welfare. According to Axiom 2, a social planner would behave by evaluating more the income change of the initially poor individuals; thus, the income change of these individuals acts more in increasing social welfare than the same income change experienced by initially richer individuals. At each given p, individuals belonging to the initial quantile group i are judged more deserving than those belonging to initial quantile group $i+1$. Note however that, even if, the relative position in each subgroup is an argument of the weighting function, no concern is expressed with respect to horizontal equity, that is how individuals in a similar initial economic condition are affected by the growth process considered.

Let $\mathbf{V_{12}} = \{< v_1(p), ..., v_m(p) >$: Axioms 1 and 2 hold$\}$ and let $\mathbf{W}_{12}$ be the class of YSWFs constructed as in (2) and based on weights profiles in $\mathbf{V}_{12}$.

The following axiom states the social relevance of inequality among equals.

Axiom 3 (Positional Transfer sensitivity) For all $p \in [0,1]$, for all $i = 1, ..., m-1$

$$v_{i-1}(p) - v_i(p) \geq v_i(p) - v_{i+1}(p)$$

This axiom states that the difference in the weight given to individuals' income changes, ranked the same in different quantile group distributions, is decreasing with the initial rank.

Let $V_{\mathbf{123}} = \{< v_1(p), ..., v_m(p) >$: Axioms 1, 2 and 3 hold$\}$ and let W_{123} be the class of YSWFs constructed as in (2) and based on weights profiles in V_{123}.

Axiom 4 (Horizontal equity) For all $p \in [0,1]$, for all $i = 1, ..., m$

$$v_i'(p) \leq 0$$

This axiom states that a social planner adopting this preference is adverse to the inequality of the growth experienced by individuals in the same quantile group.

Let $V_{\mathbf{14}} = \{< v_1(p), ..., v_m(p) >$: Axioms 1 and 4 hold$\}$ and let W_{14} be the class of YSWFs constructed as in (2) and based on weights profiles in V_{14}.

Axiom 5 (Por-poor horizontal equity) *For all* $p \in [0,1]$*, for all* $i = 1, ..., m-1$

$$v_i^{'}(p) \leq v_{i+1}^{'}(p) \leq 0$$

This axiom combines the joint effect of belonging to different initial income quantile groups and experiencing different ranks in the distribution of individual income change[6]. In this case, the marginal effect of a progressive transfer is higher the lower is the initial quantile group. That is, social welfare increases more, the lower is the initial quantile group, within which the progressive transfer of income change takes place.

Let $\mathbf{V_{1245}} = \{< v_1(p), ..., v_m(p) >$: Axioms 1,2, 4 and 5 hold$\}$ and let $\mathbf{W}_{1245}$ be the class of YSWFs constructed as in (2) and based on weights profiles in $\mathbf{V}_{1245}$.

Axiom 6 (Positional transfer sensitivity with HI aversion) *For all* $p \in [0,1]$*, for all* $i = 1, ..., m-1$

$$v_{i-1}^{'}(p) - v_i^{'}(p) \leq v_i^{'}(p) - v_{i+1}^{'}(p) \leq 0$$

This axiom combines the joint effect of horizontal inequality aversion and between group positional transfer sensitivity. It states not only that the marginal effect of a progressive transfer is higher the lower is the initial quantile group, but also that this marginal effect increases at higher pace the poorer is the quantile group considered.

[6] Note the similarty with the assumption 3b in Jenkins and Lambert (1993). See also Atkinson and Bourguignon (1987) for a similar assumption in the utilitarian framework.

Let $\mathbf{V_{1236}} = \{< v_1(p), ..., v_m(p) >$: Axioms 1,2,3, 4, 5 and 6 hold$\}$ and let $\mathbf{W}_{1236}$ be the class of YSWFs constructed as in (2) and based on weights profiles in $\mathbf{V}_{1236}$.

Axiom 7 (Horizontal Inequality neutrality) For all $p \in [0,1]$, for all $i = 1, ..., m$, $\exists \beta_i \in \Re$ such that

$$v_i(p) = \beta_i$$

This axiom states that social welfare is neutral to the inequality in each quantile group distribution of income change. Therefore, a social planner adopting this preference would give the same weight to the income change of each individual in the same quantile group.

Let $\mathbf{V_{1237}} = \{< v_1(p), ..., v_m(p) >$: Axioms 1,2,3 and 7 hold$\}$ and let $\mathbf{W}_{1237}$ be the class of YSWFs constructed as in (2) and based on weights profiles in $\mathbf{V}_{1237}$.

3 Results

In this section we discuss the dominance conditions corresponding to different classes of YSWF, depending on the properties they satisfy, all the proofs are gathered in the appendix.

We start considering the class of YSWFs that satisfy the pro-growth axiom[7].

Proposition 1 *For all growth paths* $F_A(x \mid y)$ and $F_B(x \mid y) \in D$, $W_A \geq W_B, \forall W \in \mathbf{W}_1$ *if and only if*

$$F_{Ai}^{-1}(p) \geq F_{Bi}^{-1}(p) \, \forall i = 1, ..., m, \forall p \in [0,1] \tag{3}$$

The condition expressed in eq. (3) is a first order dominance and requires that, given two within initial quantile group specific distributions of income change, $F_{Ai}(x)$ and $F_{Bi}(x)$, for all $i = 1, ..., m$, the inverse of $F_{Ai}(x)$, or quantile function, $F_{Ai}^{-1}(p)$, to lie nowhere below the inverse of $F_{Bi}(x)$, that is $F_{Bi}^{-1}(p)$, for each initial quantile group considered. When we only impose monotonicity, to understand which growth path leads to higher welfare one needs to check that, given an initial quantile group, each individual in that group shows higher income change, and this condition must be checked for every initial quantile group. This class of YSWFs is expression of a simple efficiency-based criterion, no concern is expressed towards inequality.

We now turn to the family of YSWFs that satisfy axiom 1 and 2.

Proposition 2 *For all growth paths* $F_A(x \mid y), F_B(x \mid y) \in D$, $W_A \geq W_B, \forall W \in \mathbf{W}_{12}$ *if and only if*

$$\sum_{i=1}^{k} F_{Ai}^{-1}(p) \geq \sum_{i=1}^{k} F_{Bi}^{-1}(p), \forall k = 1, ..., m, \forall p \in [0,1] \tag{4}$$

[7]For the proof of proposition 1, 2, and 3 we follow in part Peragine (2002), for proposition 5 we follow Zoli (2000).

The condition expressed in eq. (4) is a first order sequential distributional test, to be checked on the initial quantile group specific distribution of income change, starting from the lowest initial quantile, then adding the second, then the third, and so on. The condition to be satisfied at each stage is a standard first order dominance of the quantile function $F_{Ai}^{-1}(p)$ over $F_{Bi}^{-1}(p)$ of the initial quantile group specific distribution of income change. That is, the quantile function of the income change of the p poorest individuals, in terms of income change in $F_{Ai}(x)$, must be higher than the corresponding one in $F_{Bi}(x)$, and this dominance must hold for every p, by sequential aggregation of the initial quantile groups. According to eq. (4), first we have to check that, for the poorest initial quantile group, at every relative position, the dominant distribution shows higher income changes than the dominated one. Then, we have to add the second lowest initial quantile group, and so on, and perform the same check at every step, recalling that the aggregation is to be implemented on individuals ranked at the same percentile of the different quantile groups being aggregated, independently of their level of income change.

The next family of YSFWs we consider satisfies axioms 1, 2, and 3.

Proposition 3 *For all growth paths* $F_A(x \mid y), F_B(x \mid y) \in D$, $W_A \geq W_B, \forall W \in \mathbf{W}_{123}$ *if and only if*

$$\sum_{i=1}^{j} \sum_{k=1}^{i} F_{Fk}^{-1}(p) \geq \sum_{i=1}^{j} \sum_{k=1}^{i} F_{Gk}^{-1}(p), \forall j = 1, ..., m, \forall p \in [0,1] \tag{5}$$

The condition expressed in eq. (5) is a first order "sequentially cumulated" distributional test. It provides a weaker dominance condition to be applied when it is not possible to rank distributions according to proposition 1 and 2.

Proposition 4 *For all growth paths* $F_A(x \mid y), F_B(x \mid y) \in D$, $W_A \geq W_B, \forall W \in \mathbf{W}_{14}$ *if and only if*

$$\int_0^p F_{Ai}^{-1}(t) - F_{Bi}^{-1}(t)\, dt \geq 0, \forall p \in [0,1], \forall i = 1, ..., m \tag{6}$$

The condition in eq. (6) is a second order distributional test to be applied on each within group distribution of income change independently of each other. It can be represented in terms of an **income change curve (IC)**[8]. An income change curve is obtained by cumulating the quantile functions of a given initial quantile group specific income change distribution. In our framework, an initial quantile group specific income change curve can be represented putting on the x-axis the p, expressing the relative position in the initial quantile group specific distribution of income change, and on the y-axis the cumulated quantile functions; thus at $p = 1$, IC is equal to the mean income change of the individuals belonging to a given initial quantile group. The behaviour of this curve is different from the standard Generalized Lorenz curve, since it is expressed on distributions of variables that can take negative value. The main difference is

[8] See Bourguignon (2010) for a similar definition.

that, although starting from 0, the curve can take negative values and therefore being decreasing up to the last individual that experiences an income loss, which represents the "break-even point" (Lambert, 2001); from this point on the curve becomes increasing, expressing the fact that, from that part of the distribution on, individuals experience income gains. The point where the curve crosses the x-axis from below is the point where the income gains compensate for the income losses. When the mean income change is negative then the IC curve will lie in the IV quadrant. It is clear that when the variables are all positive, the IC is equivalent to the Generalized Lorenz curve applied on distributions of individual income change.

Proposition 5 *For all growth paths* $F_A(x \mid y), F_B(x \mid y) \in D$, $W_A \geq W_B, \forall W \in \mathbf{W}_{1245}$ *if and only if*

$$\sum_{j=1}^{i} \int_{0}^{p} F_{Aj}^{-1}(t)\, dt \geq \sum_{j=1}^{i} \int_{0}^{p} F_{Bj}^{-1}(t)\, dt, \forall i = 1, ..., m, \forall p \in [0,1] \tag{7}$$

The condition expressed in eq. (7) is a second order sequential distributional test, to be checked starting from the poorest initial quantile group, then adding the second, then the third, and so on. The condition to be satisfied at each stage is that the cumulated sum of the differences between the quantile functions of the two distributions of individual income change be positive[9]. According to eq. (7), first we have to check that, for the poorest initial quantile, the cumulated income change of the p poorest individuals, in terms of income change, in $F_{Ai}(x)$ is higher than the corresponding one in $F_{Bi}(x)$, and this dominance must hold for all p, where the income changes are expressed in terms of quantile function. Then, we have to add the second lowest initial quantile, and so on, and perform the same check at every step.

The condition in eq. (7) is equivalent to a convex combination of IC Curves, which is the consequence of aggregating percentiles and not levels of income change, as in the standard utilitarian framework. As a result, eq. (7) can be written as:

$$\sum_{j=1}^{i} IC_{Aj}(p) \geq \sum_{j=1}^{i} IC_{Bj}(p), \forall i = 1, ..., m, \forall p \in [0,1] \tag{8}$$

This means that we have to compare at every relative position p, of the within initial quantile distribution of income change, and at every stage i, an average of IC of all initial quantile groups not ranked higher than i. Note that,

[9] Note that, when $F^{-1}(p)$ takes negative values, $\int_0^p F^{-1}(t)\, dt$ does not measure anymore the area under the curve F. However $\int_0^p F_A^{-1}(t) - F_B^{-1}(t)\, dt$ measures the area between the two curves up tp the point p, therefore their cumulated difference can be used to make distributional comparisons.

while in Bourguignon (2010) the procedure is sequential and requires to compare sequentially the income change curves, expressed on income change level, in our framework, the dominance requires a linear combination of IC[10].

An interesting result is represented by the special case in which each initial quantile group corresponds to a single individual. In this case, the results in eq. (4) and (7) are equivalent to a IC dominance applied on distributions of income change of individuals ranked according to their position in the initial distribution of income:

$$\sum_{i=1}^{k} F_{Ai}^{-1}(1) \geq \sum_{i=1}^{k} F_{Bi}^{-1}(1) \Longleftrightarrow IC_A\left(\frac{k}{m}\right) \geq IC_B\left(\frac{k}{m}\right), \forall k = 1, ..., m \tag{9}$$

Proposition 6 *For all growth paths* $F_A(x \mid y), F_B(x \mid y) \in D$, $W_A \geq W_B, \forall W \in \mathbf{W}_{12456}$ *if and only if*

$$\sum_{k=1}^{i} \sum_{j=1}^{k} \int_{0}^{p} S_j(t)\, dt \geq 0, \forall i = 1, ..., m, \forall p \in [0, 1] \tag{10}$$

The condition expressed in eq. (10) is a second order "sequentially cumulated" distributional test. It provides a weaker dominance condition to be applied when it is not possible to rank distributions according to proposition 1 to 5.

Proposition 7 *For all growth paths* $F_A(x \mid y), F_B(x \mid y) \in D$, $W_A \geq W_B, \forall W \in \mathbf{W}_{17}$ *if and only if*[11]

$$\mu_{Fi} \geq \mu_{Gi}, \forall i = 1, ..., m \tag{11}$$

Eq. (11) is a first order direct dominance condition, or rank dominance, to be applied on distributions of mean income changes. That is, we have to check that each initial quantile group shows higher mean income growth in $F_A(x \mid y)$ than in $F_B(x \mid y)$. Thus, eq. (11) is only expression of an efficiency condition.

Proposition 8 *For all growth paths* $F_A(x \mid y), F_B(x \mid y) \in D$, $W_A \geq W_B, \forall W \in \mathbf{W}_{127}$ *if and only if*[12]

$$\sum_{i=1}^{k} \mu_{Fi} \geq \sum_{i=1}^{k} \mu_{Gi}, \forall k = 1, ..., m \tag{12}$$

[10] Bourguignon (2010), adopting a utilitarian framework, obtains a sequential generalized Lorenz dominance conditions applied on distributions of income change, thus at the final stage this is equivalent to a generalized Lorenz dominance for the income change of the whole population. Whereas, in our framework, the final stage correspond to an average over all subgroups of the generalized Lorenz curves applied on initial quantile specific distributions of individual income change.

[11] This is the dominance result derived in Van Kerm (2006) and Jenkins and Van Kerm (2011), that they call mobility profile dominance.

[12] This is the dominance result derived in Van Kerm (2006) and Jenkins and Van Kerm (2011), that they call cumulated mobility profile dominance.

Eq. (12) is a second order direct dominance to be applied on distributions of mean income changes. That is, we have to check that the cumulated sum of the initial quantile group specific mean income growth in higher in $F_A(x \mid y)$ than in $F_B(x \mid y)$. This condition is equivalent to the Generalized Lorenz dominance applied on distributions of quantile specific mean income change.

Proposition 9 *For all growth paths* $F_A(x \mid y), F_B(x \mid y) \in D$, $W_A \geq W_B, \forall W \in \mathbf{W}_{1237}$ *if and only if*[13]

$$\sum_{i=1}^{j}\sum_{k=1}^{i}\mu_{Fk} \geq \sum_{i=1}^{j}\sum_{k=1}^{i}\mu_{Gk}, \forall j = 1, ..., m \tag{13}$$

Eq. (13) is a third order direct dominance to be applied on distributions of mean income changes. It is a weaker condition, allowing to order distributions when it is not possible to rank them according to proposition 7 and 8.

3.1 Welfare dominance and na-GIC

In this section we study the relationship between the dominance conditions stated above and the na-GIC. Before analyzing these relationships, it is useful to recall the basic results on na-GIC. The procedure used to derive na-GIC consists in associating to every quantile group in the initial income distribution, the terminal income of the individual units in that quantile group. Formally, Bourguignon (2010) defines na-GIC as follows:

$$\tilde{g}_i = \frac{\int_0^a w dF_i(w)}{y_i} - 1, \text{ for all } i = 1, ..., m \tag{14}$$

Eq. (14) states that the na-GIC measures the mean income growth of all individuals belonging to a given quantile group in the initial distribution of income, independently of the final quantile group they belong to. The first order dominance, therefore, consists in comparing $\tilde{g}_i$ for every quantile of the initial distribution of income, $i = 1, ..., m$.

Let us start with the dominance condition, obtained by considering the family of YSWFs $W \in \mathbf{W}_1$ and $W \in \mathbf{W}_{14}$. Note that if the dominance in eq. (3) and in eq. (6) holds for every p, than it must be the case that it holds for $p = 1$. Integrating the quantile function from $p = 0$ up to $p = 1$, for all p belonging to the same initial quantile, the social welfare dominance implies that:

$$\int_0^1 F_{Ai}^{-1}(p)\, dp \geq \int_0^1 F_{Bi}^{-1}(p)\, dp, \forall i = 1, ..., m \Longleftrightarrow$$

$$\mu_{iA} \geq \mu_{iB}, \forall i = 1, ..., m$$

[13] This is the "Type C" dominance result derived in Van Kerm (2006).

where $\mu_{Ai} = \int_0^1 F_{Ai}^{-1}(p)\,dp$ is the mean income growth, expressed in absolute terms, of the individuals belonging to the same initial quantile, which is also equal to $\int_0^{\hat{x}} x dF_i(x)$, where recall that $x = w - y$.

Moreover, note that $\tilde{g}_i$ can also be written in terms of quantile functions: $\tilde{g}_i = \frac{\int_0^1 F_i^{-1}(p)dp}{y_i}$ implying $\mu_{Ai} = \tilde{g}_{Ai} y_{Ai}$.

Thus

$$W^A \geq W^B \Longrightarrow \mu_{Ai} \geq \mu_{Bi}, \forall i = 1, ..., m$$

means that

$$W^A \geq W^B \Longrightarrow \tilde{g}_{Ai} y_{Ai} \geq \tilde{g}_{Bi} y_{Bi}, \forall i = 1, ..., m$$

If we use relative measures of income change, we can also write the following:

$$W^A \geq W^B \Longrightarrow \tilde{g}_{Ai} \geq \tilde{g}_{Bi}, \forall i = 1, ..., m \tag{15}$$

We can summarize the previous discussion in the following proposition.

Proposition 10 *For all growth paths $F_A(x \mid y), F_B(x \mid y) \in D$, if $W_A \geq W_B$, $\forall W \in \mathbf{W}_1$ and $\forall W \in \mathbf{W}_{14}$ then*

$$\tilde{g}_{Ai} y_{Ai} \geq \tilde{g}_{Bi} y_{Bi}, \forall i = 1, ..., m \tag{16}$$

According to proposition 10, first order dominance of na-GIC, weighted by the initial income of each quantile, is a necessary condition for social welfare dominance, evaluated along two given growth paths. Of course, the condition of a "weighted" na-GIC dominance depends on the fact that we have imposed $x = w - y$. If we use a relative measure, this dominance would reduce to the first order na-GIC dominance.

It is obvious that when we impose, in addition to axiom 1 and 4, axiom 7, the same implication holds, with the difference that if we consider relative measure of income change we get an equivalence result.

We now turn to the dominance condition[14] obtained by considering the family of YSWFs satisfying also axiom 1, 2, 4, and 5. Note that if the dominance in eq. (4) and in eq. (7) holds for every p, than it must be the case that it holds for $p = 1$. Integrating sequentially the inverse cumulative distribution function up to $p = 1$, the social welfare dominance implies that

$$\hat{\mu}_k \left[F_A^{-1}(p)\right] \geq \hat{\mu}_k \left[F_B^{-1}(p)\right] \tag{17}$$

[14] We do not consider the case of YSWF satisfying axioms 1 and 2 since the implications in terms of naGIC are the same as the YSWF satisfying axioms 1, 2 and 5.

where $\hat{\mu}_k\left[F_A^{-1}(p)\right] = \sum_{i=1}^{k}\mu_{Ai}$, $\forall k = 1,...,m, \forall p \in [0,1]$, can be considered as sequential averages of the total distribution. That is, a sequential weighted mean income change dominance. Social welfare dominance implies that at every stage we have to evaluate that the mean income growth is higher in one growth path than in the other. The procedure to follow is: first, take the poorest initial quantile group, evaluate the mean income growth of the individuals belonging to it, and check the dominance with respect to the alternative growth path. Then add the second, aggregate each individual income growth of the individuals belonging to the same percentile in the two poorest initial quantile groups, apply the overall average income growth over this new distributions, and check the dominance with respect to the alternative growth path. Then, add the third and so on, and apply the same steps as above.

Note that $\sum_{i=1}^{k}\mu_{Ai} = \sum_{i=1}^{k}\tilde{g}_{Ai}y_{Ai}$, thus

$$W^A \geq W^B \Longrightarrow \sum_{i=1}^{k}\tilde{g}_{Ai}y_{Ai} \geq \sum_{i=1}^{k}\tilde{g}_{Bi}y_{Bi}, \forall k = 1,...,m \tag{18}$$

Therefore, if we use a relative measure of income change, the following relationship holds:

$$W^A \geq W^B \Longrightarrow \sum_{i=1}^{k}\tilde{g}_{Ai} \geq \sum_{i=1}^{k}\tilde{g}_{Bi}, \forall i = 1,...,m \tag{19}$$

We can summarize the previous discussion in the following proposition.

Proposition 11 *For all growth paths* $F_A(x \mid y), F_B(x \mid y) \in D$, *if* $W_A \geq W_B$, $\forall W \in \mathbf{W}_{12}$ and $\forall W \in W_{1245}$ *then*

$$\sum_{i=1}^{k}\tilde{g}_{Ai}y_{Ai} \geq \sum_{i=1}^{k}\tilde{g}_{Bi}y_{Bi}, \forall k = 1,...,m \tag{20}$$

The condition in eq. (20) can be interpreted as a particular version of the cumulated na-GIC. According to proposition 11, second order dominance (or cumulated) of na-GIC weighted by the initial income of each quantile group is a necessary condition for social welfare dominance, evaluated along two given growth paths. As before, the presence of the weights depends on the fact that we have expressed our income change function as the absolute income change. If we use a relative measure, this dominance would reduce to the second order na-GIC dominance. As a result, we can state that the second order initial income weighted na-GIC dominance is a necessary condition for social welfare dominance evaluated along two given growth paths, where the YSWF satisfies axiom 1, 2 and 1, 2, 4 and 5.

A special case is given by distributions where each initial quantile group corresponds to one individual. In this case the first order dominance of proposition 1 is equivalent to the initial income weighted na-GIC dominance.

$$W^A \geq W^B,\ \forall W \in \mathbf{W}_1 \Longleftrightarrow$$

$$F_{Ai}^{-1}(1) \geq F_{Bi}^{-1}(1) \Longleftrightarrow x_{Ai} \geq x_{Bi}, \forall k = 1, ..., m$$

and recall that: $x_i = w_i - y_i$ and $x_i = \tilde{g}_i y_i$. The following relationship holds:

$$W^A \geq W^B,\ \forall W \in \mathbf{W}_1 \Longleftrightarrow \tilde{g}_{Ai} y_{Ai} \geq \tilde{g}_{Bi} y_{Bi},\ \forall i = 1, ..., m \tag{21}$$

Thus, ranking distributions according to a YSWF satisfying axiom 1 is equivalent to ranking them according to the initial income weighted na-GIC.

The same reasoning can be done for first and second order sequential dominance. When each quantile encompasses only one individual the dominance conditions in Proposition 2 and 5 are equivalent to cumulated initial income weighted na-GIC.

$$W^A \geq W^B,\ \forall W \in \mathbf{W}_{125} \Longleftrightarrow$$

$$\sum_{i=1}^{k} F_{Ai}^{-1}(1) \geq \sum_{i=1}^{k} F_{Bi}^{-1}(1) \Longleftrightarrow \sum_{i=1}^{k} x_{Ai} \geq \sum_{i=1}^{k} x_{Bi}, \forall k = 1, ..., m$$

and recall that: $x_i = w_i - y_i$ and $x_i = \tilde{g}_i y_i$. The following relationship holds:

$$W^A \geq W^B,\ \forall W \in \mathbf{W}_{125} \Longleftrightarrow \sum_{i=1}^{k} \tilde{g}_{Ai} y_{Ai} \geq \sum_{i=1}^{k} \tilde{g}_{Bi} y_{Bi},\ \forall k = 1, ..., m \tag{22}$$

Thus, ranking distributions according to a YSWF satisfying axiom 1, 2 and axioms 1,2, 4 and 5, is equivalent to ranking them according to the cumulated initial income weighted[15] na-GIC.

3.1.1 A measure of the horizontal inequality of growth

The aim of this section is to propose a measure of the horizontal inequality of growth, that is how growth affects differently individuals in similar initial economic conditions.

Let J_i be a Gini-type measure of inequality applied to initial quantile specific distributions of income change. As a results, J_i can be interpreted as a measure of horizontal inequality (HI) of growth among individuals having the same relative position in the initial income distribution. Then, for a given $v_i(p)$, such that $v_i(p)$ is non-increasing with the rank p, and $\int_0^1 v_i(p)\, dp = 1$, J_i can be defined by:

[15] If we further impose axiom 4, the same implication holds.

$$J_i = 1 - \frac{\int_0^1 v_i(p) F_i^{-1}(p)\, dp}{\mu_i} \tag{23}$$

In this case, the mean income growth is corrected by J_A, a Gini-type inequality index, measuring the horizontal inequality of growth.

However note that this index might not be define when the mean income change is equal to 0, that is no growth has taken place. This problem may be solved by using the absolute version of the Gini-type index of HI. It can be obtained by multiplying J_i by the mean income changes and it can be defined as follows:

$$AJ_i = \mu_i \left(1 - \frac{\int_0^1 v_i(p) F_i^{-1}(p)\, dp}{\mu_i} \right) = \mu_i - \int_0^1 v_i(p) F_i^{-1}(p)\, dp \tag{24}$$

There is horizontal equity when $AJ_i = 0$. By contrast, the higher is the distance[16] from 0 the more is the horizontal inequality of growth.

An overall index of HI can be defined by simple aggregation:

$$AJ = \sum_{i=1}^{m} J_i \tag{25}$$

3.1.2 A measure of the progressivity of growth

The aim of this section is to propose an index of vertical inequality. For, we star from a social welfare function which satisfies Horizontal inequality neutrality, such that it can also we written as follows:

$$W(F(x \mid y)) = \sum_{i=1}^{m} \int_0^1 v_i(p) F_i^{-1}(p)\, dp = \tag{26}$$

$$= \sum_{i=1}^{m} v_i \int_0^1 F_i^{-1}(p)\, dp = \sum_{i=1}^{m} q_i v_i \mu_i$$

where $\mu_i = \int_0^1 F_i^{-1}(p)\, dp$ and v_i is non-increasing with i.

[16]We use the term distance since the index can take both positive and negative value, respectively when the mean income growth of the group considered is positive and when it is negative.

This kind of social welfare function builds on a distribution summarizing the extent of growth for each initial quantile. In particular, this distribution is represented by the mean income growth experienced by each quantile of the initial distribution. On this distribution we can apply an index of vertical inequality (or progressivity) defined as follows:

$$VI = \frac{\sum_{i=1}^{m} v_i \mu_i}{\mu} \tag{27}$$

where $\mu = \sum_{i=1}^{m} \mu_i$ and v satisfies the normalization condition[17] $\sum_{i=1}^{m} v_i = 1$. VI measures the relative distance between our actual social welfare evaluation of growth, obtained through the use of a YSWF[18], where more weight is given to the income growth experienced by initially poorer individuals, and the income growth each initial quantiles would experience in the case of a proportional growth. $VI > 1$ means that we are in presence of a progressive growth path, that is growth is concentrated more among individuals ranked lower in the initial distribution of income; $VI < 1$ means that the growth path is regressive, that is income growth is concentrated more among the initially rich individuals; $VI = 1$ means that every individual experiences a proportional growth.

A last remark is to show that progressivity of growth is social welfare improving. It is widespread in the literature the perception of social welfare as a trade-off between inequality and efficiency[19], which arises to be meaningful in terms of complete ordering of distributions. Different contributions (Lambert, 2001; Aaberge, 2001) show that social welfare admits a decomposition with respect to average income and inequality.

A similar formulation for social welfare can be derived under the light of the social welfare evaluation of growth:

$$W = \mu \left(1 - J^*\right) \tag{28}$$

where $J^* = 1 - VI$.

Thus, we have a social welfare decomposition into a term expressing the overall extent of growth and a term expressing the progressivity of growth. Given the value that the index of progressivity can take, it is easy to see that when growth is pro-poor, ($VI > 1$), the abbreviated social welfare increases, whereas when growth is pro-rich ($VI < 1$), social welfare decreases.

[17] In alternative we could write the index as follows: $VI = \frac{\sum_{i=1}^{m} q_i v_i \mu_i}{\mu \sum_{i=1}^{m} q_i v_i}$.

[18] A special case of this measure is provided by Jenkins and Van Kerm (2011; 2009) and Van Kerm (2006).

[19] See Lambert (2001) for an extensive discussion on this topic.

4 Conclusions

Recent contributions have sustained the need to modify standard frameworks for measuring the distributional effect of growth, in order to take into account the possibility of reshuffling of individuals among income classes. In this chapter we have provided a normative approach to rank growth paths when these further aspects are a matter of concern. For, we have adopted a bi-dimensional framework, where the two dimensions are respectively the initial economic condition of individuals and his income transformation. We have proposed to aggregate these information according to a rank dependent approach, which makes it possible to account for the identity of individuals and their movement along the distribution. We have provided partial dominance conditions for ordering growth paths and we have shown how these conditions relate to na-GIC. We have provided partial dominance conditions for ordering growth processes and we have shown how these conditions relate to na-GIC. Finally, we have proposed two classes of indices: one aimed at capturing the extent of the inequality of growth; the other aimed at capturing the progressivity of growth.

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5 Appendix

Proof of Proposition 1

We want to find a necessary and sufficient condition for

$$\Delta W\left(F\left(x \mid y\right)\right)=\sum_{i=1}^{m}\int_{0}^{1} v_i\left(p\right)\left[F_{Ai}^{-1}\left(p\right)-F_{Bi}^{-1}\left(p\right)\right]dp\geq 0,\ \forall W\in \mathbf{W}_1 \qquad (29)$$

Sufficiency clearly derives from the fact that since $v_i\left(p\right)\geq 0$, $\forall p\in[0,1]$ and $\forall i=1,2,...,m$, $F_{Ai}^{-1}\left(p\right)-F_{Ai}^{-1}\left(p\right)\geq 0$, $\forall p\in[0,1]$ and $\forall i=1,2,...,m$, implies $\Delta W\left(F\left(x\mid y\right)\right)\geq 0$.

For the necessity, suppose for a contradiction that $\Delta W\left(F\left(x\mid y\right)\right)\geq 0, \forall W\in \mathbf{W}_1$, but there is a quantile group $h\in\{1,...,m\}$ and an interval $I\equiv[a,b]\subseteq[0,1]$ such that $F_{Ah}^{-1}\left(p\right)-F_{Bh}^{-1}\left(p\right)<0, \forall p\in I$. Now select a set of function $\{v_i\left(p\right)\}_{i\in\{1,...,m\}}$ such that $v_i\left(p\right)\searrow 0$, $\forall i\neq h$ and $v_h\left(p\right)\searrow 0$, $\forall p\in[0,1]\setminus I$. in this case $\Delta W\left(F\left(x\mid y\right)\right)$ would reduce to $\int\limits_{a}^{b} v_h\left(p\right)\left[F_{Ah}^{-1}\left(p\right)-F_{Bh}^{-1}\left(p\right)\right]dp<0$, a contradiction. **QED**

Proof of Proposition 2

Before proving this proposition we need to prove the following lemma.

Lemma 1 $\sum_{k=1}^{m} v_k w_k \geq 0$ *for all sets of numbers* $\{v_k\}$ *such that* $v_k \geq v_{k+1} \geq 0$ $\forall k \in \{1,...,m\}$ *if and only if* $\sum_{i=1}^{k} w_i \geq 0$, $\forall k \in \{1,...,m\}$.

Proof. Applying Abel's decomposition: $\sum_{k=1}^{m} v_k w_k = \sum_{k=1}^{m}\left(v_k - v_{k+1}\right)\sum_{i=1}^{k} w_i$. If $\sum_{i=1}^{k} w_i \geq 0$, $\forall k \in \{1,...,m\}$, then $\sum_{k=1}^{m} v_k w_k \geq 0$. As for the necessity part, suppose that $\sum_{k=1}^{m} v_k w_k \geq 0$ for all sets of numbers $\{v_k\}$ such that $v_k \geq v_{k+1} \geq 0$, but $\exists j \in \{1,...,m\} : \sum_{i=1}^{j} w_i < 0$. Consider what happens when $\left(v_k - v_{k+1}\right) \searrow 0, \forall k \neq j$. We obtain that $\sum_{k=1}^{m} v_k w_k \longrightarrow \left(v_j - v_{j+1}\right)\sum_{i=1}^{j} w_i < 0$, a contradiction. ■

We now want to find a necessary and sufficient condition for

$$\Delta W\left(F\left(x\mid y\right)\right)=\sum_{i=1}^{m}\int\limits_{0}^{1} v_i\left(p\right)\left[F_{Ai}^{-1}\left(p\right)-F_{Bi}^{-1}\left(p\right)\right]dp\geq 0,\ \forall W\in\mathbf{W}_{12} \qquad (30)$$

Sufficiency can be shown as follows. First, reverse the order of integration and summation, such that

$$\Delta W\left(F\left(x\mid y\right)\right)=\int\limits_{0}^{1}\sum_{i=1}^{m} v_i\left(p\right)\left[F_{Ai}^{-1}\left(p\right)-F_{Bi}^{-1}\left(p\right)\right]dp\geq 0 \qquad (31)$$

Letting $S_i\left(p\right)=F_{Ai}^{-1}\left(p\right)-F_{Bi}^{-1}\left(p\right)$ and rewriting (42):

$$\Delta W\left(F\left(x\mid y\right)\right)=\int\limits_{0}^{1}\sum_{i=1}^{m} v_i\left(p\right)S_i\left(p\right)dp\geq 0 \qquad (32)$$

Since $v_i(p) \geq v_{i+1}(p) \geq 0, \forall i = 1, ..., m-1$ and $\forall p \in [0,1]$, we can apply Lemma 1 and obtain that $\sum_{i=1}^{m} v_i(p) S_i(p) \geq 0$ if and only if $\sum_{i=1}^{k} S_i(p) \geq 0, \forall k = 1, ..., m$ and $\forall p \in [0,1]$. It follows that $\sum_{i=1}^{m} v_i(p) S_i(p) \geq 0$, $\forall p \in [0,1]$, implies that, integrating with respect to p, $\int_0^1 \sum_{i=1}^{m} v_i(p) S_i(p)\, dp \geq 0$.

For the necessity, suppose for a contradiction that $\Delta W(F(x \mid y)) \geq 0, \forall W \in \mathbf{W}_{12}$, but there is an initial quantile $h \in \{1, ..., m\}$ and an interval $I \equiv [a,b] \subseteq [0,1]$ such that $\sum_{i=1}^{h} S_i(p) < 0, \forall p \in I$. Now, applying Lemma 1, there exists a set of functions $\{v_i(p) \geq 0\} : [0,1] \longrightarrow \Re_+$, $i = 1, ..., m$, such that $\sum_{i=1}^{m} v_i(p) S_i(p) < 0, \forall p \in I$. Writing $\sum_{i=1}^{m} v_i(p) S_i(p) = T(p)$, $\Delta W(F(x \mid y))$ reduces to $\int_0^1 T(p)\, dp$, where $T(p) < 0$, $\forall p \in I$. Selecting a set of function $T(p)$, such that $T(p) \longrightarrow 0$, $\forall p \in [0,1] \setminus I$, $\Delta W(F(x \mid y))$ would reduce to $\int_a^b T(p)\, dp < 0$, a contradiction.
QED

Proof of Proposition 3

We want to find a necessary and sufficient condition for

$$\Delta W(F(x \mid y)) = \sum_{i=1}^{m} \int_0^1 v_i(p) \left[F_{Ai}^{-1}(p) - F_{Bi}^{-1}(p)\right] dp \geq 0, \forall W \in \mathbf{W}_{123} \quad (33)$$

For the sufficiency, note that if $v_i(p)$ satisfies axiom 1, 2, and 3, we can revert the order of integration and summation and apply Abel's decomposition to obtain: $\Delta W(F(x \mid y)) = \int_0^1 \left[\sum_{i=1}^{n} (v_i(p) - v_{i+1}(p)) \sum_{k=1}^{i} S_k(p)\right] dp$, where $S_k(p) = F_{Ak}^{-1}(p) - F_{Bk}^{-1}(p)$. Let $v_i(p) - v_{i+1}(p) = \omega_i(p)$ and $\sum_{k=1}^{i} S_k(p) = \kappa_i(p)$, by axiom 3 $\omega_i(p) > \omega_{i+1}(p)$, $\forall i = 1, ..., m-1$, $\forall p \in [0,1]$. We can apply Lemma 1 to get $\sum_{i=1}^{m} \omega_i(p) \kappa_i(p) \geq 0$ if and only if $\sum_{i=1}^{j} \kappa_i(p) \geq 0$, $\forall j = 1, ..., m$, $\forall p \in [0,1]$. If $\sum_{i=1}^{m} \omega_i(p) \kappa_i(p) \geq 0$, $\forall p \in [0,1]$, implies that $\Delta W(F(x \mid y)) =$

$\int_0^1 \sum_{i=1}^{m} \omega_i(p)\,\kappa_i(p)\,dp \geq 0$. Thus $\sum_{i=1}^{j}\sum_{k=1}^{i} F_{Ai}^{-1}(p) - F_{Bi}^{-1}(p)$, $\forall j = 1,...,m$, $\forall p \in [0,1]$ is sufficient for $\Delta W(F(x \mid y)) \geq 0$.

For the necessity, let $T(p) \equiv \sum_{i=1}^{m} \omega_i(p)\,\kappa_i(p)$, we can write the following $\Delta W(F(x \mid y)) = \int_0^1 T(p)\,dp$. Suppose that $\Delta W(F(x \mid y)) \geq 0$, $\forall W \in \mathbf{W}_{123}$, but $\exists h = 1,...,m$ and $\exists I \equiv [a,b] \subseteq [0,1]$ such that $\sum_{i=1}^{h} \kappa_i(p) < 0$, $\forall p \in I$. Then by Lemma 1 $\exists$ a set of functions $\omega_i(p) : [0,1] \longrightarrow \Re_+$, $i = 1,...,m$, such that $T(p) < 0$, $\forall p \in I$. We can chose a function $T(p)$ such that $T(p) \longrightarrow 0$, $\forall p \in [0,1] \diagdown I$, thus $\Delta W(F(x \mid y))$ would reduce to $\int_a^b T(p)\,dp < 0$, a contradiction.

QED

Proof of Proposition 4

We want to find a necessary and sufficient condition for

$$\Delta W(F(x \mid y)) = \sum_{i=1}^{m} \int_0^1 v_i(p)\left[F_{Ai}^{-1}(p) - F_{Bi}^{-1}(p)\right] dp \geq 0, \forall W \in \mathbf{W}_{14} \quad (34)$$

For the sufficiency, let $S_i(p) = F_{Ai}^{-1}(p) - F_{Bi}^{-1}(p)$, we can integrate eq. (45) by parts to get:

$$\Delta W(F(x \mid y)) = \sum_{i=1}^{m}\left[v_i(1)\int_0^1 S_i(p)\,dp\right] - \sum_{i=1}^{m}\int_0^1 v_i'(p)\int_0^p S_i(t)\,dt\,dp \quad (35)$$

It follows that

$$\int_0^p F_{Ai}^{-1}(t) - F_{Bi}^{-1}(t)\,dt \geq 0, \forall p \in [0,1], \forall i = 1,...,m \quad (36)$$

is sufficient for welfare dominance, since by axiom 1 $v_i(p) \geq 0$ eq. (47) implies the positivity of $v_i(1)\int_0^1 S_i(p)\,dp$ and by axiom 6 $v_i'(p) \leq 0$ it implies the negativity of the second term of eq. (47). It follows that $\Delta W(F(x \mid y)) \geq 0$.

For the necessity, suppose that $\Delta W(F(x \mid y)) \geq 0$, but $\exists h = 1,...,m$ and $\exists I \equiv [a,b] \subseteq [0,1]$ such that $\int_0^p F_{Ah}^{-1}(t) - F_{Bh}^{-1}(t)\,dt < 0$, $\forall p \in I$. Let $T_i(p) =$

$v_i'(p)\int\limits_0^p F_{Ai}^{-1}(t)-F_{Bi}^{-1}(t)\,dt,\ \forall i=1,...,m,\ \forall p\in[0,1]$. We can chose a set of functions $T_i(p)$ such that $T_i(p)\longrightarrow 0,\ \forall i\neq h$ and $T(p)\longrightarrow 0,\ \forall p\in[0,1]\diagdown I$. Given the negativity of $v_i'(p)$, $T_h(p)<0,\ \forall p\in I$. Then $\Delta W(F(x\mid y))=\sum\limits_{i=1}^m v_i(1)\int\limits_0^1 S_i(p)\,dp-\int\limits_a^b T_h(p)\,dp$. It is always possible to chose a combination of $v_i(1)$ and $S_i(p)$ such that $\sum\limits_{i=1}^m v_i(1)\int\limits_0^1 S_i(p)\,dp\leq 0$. Then, $\Delta W(F(x\mid y))=-\int\limits_a^b T_h(p)\,dp\leq 0$, a contradiction. **QED**

Proof of Proposition 5

We want to find a necessary and sufficient condition for

$$\Delta W(F(x\mid y))=\sum_{i=1}^m\int\limits_0^1 v_i(p)\left[F_{Ai}^{-1}(p)-F_{Bi}^{-1}(p)\right]dp\geq 0,\ \forall W\in\mathbf{W}_{1245} \tag{37}$$

Sufficiency can be shown as follows. First, reverse the order of integration and summation, such that

$$\Delta W(F(x\mid y))=\int\limits_0^1\sum_{i=1}^m v_i(p)\left[F_{Ai}^{-1}(p)-F_{Bi}^{-1}(p)\right]dp\geq 0 \tag{38}$$

Letting $S_i(p)=F_{Ai}^{-1}(p)-F_{Bi}^{-1}(p)$ and $\varepsilon_i(p)=v_i(p)-v_{i+1}(p)\geq 0,\ \forall i=1,...,m-1$ and $\forall p\in[0,1]$, by axiom 1 and application of Abel's decomposition we can rewrite eq. (49):

$$\Delta W(F(x\mid y))=\int\limits_0^1\sum_{i=1}^m\left(\varepsilon_i(p)\sum_{j=1}^i S_j(p)\right)dp\geq 0 \tag{39}$$

Integrating by parts

$$\Delta W(F(x\mid y))=\sum_{i=1}^m\left(\varepsilon_i(1)\sum_{j=1}^i\int\limits_0^1 S_j(p)\,dp\right)- \tag{40}$$

$$\int\limits_0^1\sum_{i=1}^m\left(\varepsilon_i^{'}(p)\sum_{j=1}^i\int\limits_0^p S_j(t)\,dt\right)dp\geq 0$$

By axiom 2 $\varepsilon_i(1) \geq 0$, by axiom 5 $\varepsilon_i^{'}(p) \leq 0$, $\forall i = 1, ..., m$ and $\forall p \in [0,1]$, it follows that $\sum_{j=1}^{i} \int_0^p S_j(t)\, dt \geq 0, \forall i = 1, ..., m$ and $\forall p \in [0,1]$ implies $\Delta W(F(x \mid y)) \geq 0$.

For the necessity, suppose for a contradiction that $\Delta W(F(x \mid y)) \geq 0$, but there is an initial quantile $h \in \{1, ..., m\}$ and an interval $I \equiv [a,b] \subseteq [0,1]$ such that $\sum_{j=1}^{h} \int_0^p S_j(t)\, dt < 0, \forall p \in I$. Now, applying Lemma 1 in Chambaz and Maurin (1998), there exists a set of non-positive functions $\left\{\varepsilon_i^{'}(p) \leq 0\right\}_{i \in \{1,...,m\}}$ such that $\sum_{i=1}^{m} \varepsilon_i^{'}(p) \left(\sum_{j=1}^{i} \int_0^p S_j(t)\, dt \right) > 0$, $\forall p \in I$.

Now let $R(p) = \sum_{i=1}^{m} \varepsilon_i^{'}(p) \left(\sum_{i=1}^{i} \int_0^p S_i(t)\, dt \right)$, then $R(p) \geq 0$, $\forall p \in I$. Then, $\Delta W(F(x \mid y)) = \sum_{i=1}^{m} \left(\varepsilon_i(1) \sum_{j=1}^{i} \int_0^1 S_j(p)\, dp \right) - \int_0^1 R(p)\, dp.$

Now choosing $R(p)$ such that $R(p) \longrightarrow 0$ for some $p \in [0,1] \setminus I$,

$$\Delta W(F(x \mid y)) = \sum_{i=1}^{m} \left(\varepsilon_i(1) \sum_{j=1}^{i} \int_0^1 S_j(p)\, dp \right) - \int_a^b R(p)\, dp$$

We can choose $\varepsilon_i(1) = 0$, $\forall i = 1, ..., m$, or we can choose a combination of $\varepsilon_i(1)$ and $\sum_{j=1}^{i} \int_0^1 S_j(p)\, dp$, such that $\sum_{i=1}^{m} \left(\varepsilon_i(1) \sum_{j=1}^{i} \int_0^1 S_j(p)\, dp \right) = 0$, then $\Delta W(F(x \mid y)) = -\int_a^b R(p)\, dp \leq 0$, a desired contradiction. **QED**.

Proof of proposition 6

We want to find necessary and sufficient conditions for

$$\Delta W = \sum_{i=1}^{m} \int_0^1 v_i(p)\, F_i^{-1}(p)\, dp - \sum_{i=1}^{m} \int_0^1 v_i(p)\, G_i^{-1}(p)\, dp \geq 0 \tag{41}$$

where $v_{i-1}'(p) - v_i'(p) < v_i'(p) - v_{i+1}'(p) < 0$.

For the sufficiency, let $S_i(p) = \Delta W = F_i^{-1}(p) - G_i^{-1}(p)$

$$\Delta W = \sum_{i=1}^{m} \int_0^1 v_i(p)\, S_i(p)\, dp \geq 0 \tag{42}$$

Integrating by parts

$$\Delta W = \sum_{i=1}^{m} v_i(1) \int_0^1 S_i(p)\,dp - \sum_{i=1}^{m} \int_0^1 v_i'(p) \int_0^p S_i(t)\,dt \geq 0 \tag{43}$$

Simplifying

$$\Delta W = \sum_{i=1}^{m} v_i(1)(\mu_{Fi} - \mu_{Gi}) - \sum_{i=1}^{m} \int_0^1 v_i'(p) \int_0^p S_i(t)\,dt = \tag{44}$$

$$= \sum_{i=1}^{m} v_i(1)(\mu_{Fi} - \mu_{Gi}) - \int_0^1 \sum_{i=1}^{m} v_i'(p) \int_0^p S_i(t)\,dt \geq 0 \tag{45}$$

By axiom 4 and 5 $v_i'(p) - v_{i+1}'(p) = w_i'(p) \leq 0$.

For analytical convenience let $w_i'(p) = -z_i(p)$, where $z_i(p) > 0$, such that $w_{i+1}'(p) = -z_{i+1}(p)$, this implies that $z_i > z_{i+1}$

$$\Delta W = \sum_{i=1}^{m} v_i(1)(\mu_{Fi} - \mu_{Gi}) + \int_0^1 \sum_{i=1}^{m} z_i(p) \sum_{k=1}^{i} \int_0^p S_k(t)\,dt = \tag{46}$$

$$= \sum_{i=1}^{m} v_i(1)(\mu_{Fi} - \mu_{Gi}) + \int_0^1 \sum_{i=1}^{m} z_i(p) - z_{i+1}(p) \sum_{k=1}^{i} \sum_{j=1}^{k} \int_0^p S_k(t)\,dt \geq 0 \tag{47}$$

By axiom 1 and 2 $v_i(p) - v_{i+1}(p) > 0$

$$\Delta W = \sum_{i=1}^{m} v_i(1) - v_{i+1}(1) \sum_{k=1}^{i} (\mu_{Fk} - \mu_{Gk}) + \int_0^1 \sum_{i=1}^{m} z_i(p) - z_{i+1}(p) \sum_{k=1}^{i} \sum_{j=1}^{k} \int_0^p S_j(t)\,dt \geq 0 \tag{48}$$

Let $v_i(p) - v_{i-1}(p) = w_{i-1}(p)$

$$\Delta W = \sum_{i=1}^{m} w_i(1) \sum_{k=1}^{i} (\mu_{Fk} - \mu_{Gk}) + \int_0^1 \sum_{i=1}^{m} z_i(p) - z_{i+1}(p) \sum_{k=1}^{i} \sum_{j=1}^{k} \int_0^p S_j(t)\,dt \geq 0 \tag{49}$$

Also, by axiom 3 $w_i(p) > w_{i+1}(p) > 0$

$$\Delta W=\sum_{i=1}^{m}w_i\left(1\right)-w_{i+1}\left(1\right)\sum_{k=1}^{i}\sum_{j=1}^{k}\left(\mu_{Fj}-\mu_{Gj}\right)+\int_{0}^{1}\sum_{i=1}^{m}z_i\left(p\right)-z_{i+1}\left(p\right)\sum_{k=1}^{i}\sum_{j=1}^{k}\int_{0}^{p}S_j\left(t\right)dt\geq 0 \tag{50}$$

It follows that

$$\sum_{k=1}^{i}\sum_{j=1}^{k}\int_{0}^{p}S_j\left(t\right)dt\geq 0,\forall i=1,...,m,\forall p\in[0,1] \tag{51}$$

is sufficient for $\Delta W\geq 0$. In fact, if eq. (62) holds, then $\sum_{k=1}^{i}\sum_{j=1}^{k}\left(\mu_{Fj}-\mu_{Gj}\right)\geq 0$, $\forall i=1,...,m$,

And if $\sum_{k=1}^{i}\sum_{j=1}^{k}\int_{0}^{p}S_j\left(t\right)dt\geq 0$, $\forall i=1,...,m$, also holds $\forall p\in[0,1]$,

then it must be the case that $\int_{0}^{1}\sum_{i=1}^{m}z_i\left(p\right)-z_{i+1}\left(p\right)\sum_{k=1}^{i}\sum_{j=1}^{k}\int_{0}^{p}S_j\left(t\right)dt\geq 0$.

For the necessity: suppose that $\Delta W\geq 0$, but $\exists h=1,...,m$ and $\exists I\in[a,b]\subseteq[0,1]$

such that $\sum_{k=1}^{h}\sum_{j=1}^{k}\int_{0}^{p}S_j\left(t\right)dt\leq 0$, $\forall p\in I$.

By lemma 1, this implies that there is a function $z_i\left(p\right)$

such that $\sum_{i=1}^{m}z_i\left(p\right)-z_{i+1}\left(p\right)\sum_{k=1}^{i}\sum_{j=1}^{k}\int_{0}^{p}S_j\left(t\right)dt\leq 0$, $\forall p\in I$.

Let $T\left(p\right)\equiv\sum_{i=1}^{m}z_i\left(p\right)-z_{i+1}\left(p\right)\sum_{k=1}^{i}\sum_{j=1}^{k}\int_{0}^{p}S_j\left(t\right)dt$, $\forall p\in[0,1]$; $\int_{a}^{b}T\left(p\right)dp<0$.

We can chose a function $T\left(p\right)\longrightarrow 0$, $\forall p\in[0,1]\diagdown I$.

This implies that $\Delta W=\sum_{i=1}^{m}w_i\left(1\right)\sum_{k=1}^{i}\left(\mu_{Fi}-\mu_{Gi}\right)+\int_{a}^{b}T\left(p\right)dp$. It is always possible to chose a combination of $w_i\left(1\right)$ and $\mu_{Fi}-\mu_{Gi}$ such that $\sum_{i=1}^{m}w_i\left(1\right)\sum_{k=1}^{i}\left(\mu_{Fi}-\mu_{Gi}\right)=0$. It follows that $\Delta W=\int_{a}^{b}T\left(p\right)dp\leq 0$ a contradiction. **QED**

Proof of Proposition 7

We want to find a necessary and sufficient condition for

$$\Delta W\left(F\left(x\mid y\right)\right)=\sum_{i=1}^{m}\int_{0}^{1}v_{i}\left(p\right)\left[F_{Ai}^{-1}\left(p\right)-F_{Bi}^{-1}\left(p\right)\right]dp\geq 0,\ \forall W\in\mathbf{W}_{17} \quad (52)$$

For the sufficiency, by axiom 7 $v_i(p)=\beta_i$, $\forall p\in[0,1]$ and $\forall i=1,2,...,m$, therefore we can write eq. (63) as follows:

$$\Delta W\left(F\left(x\mid y\right)\right)=\sum_{i=1}^{m}\beta_{i}\int_{0}^{1}\left[F_{Ai}^{-1}\left(p\right)-F_{Bi}^{-1}\left(p\right)\right]dp=$$

$$=\sum_{i=1}^{m}\beta_{i}\left[\mu_{Ai}-\mu_{Bi}\right]\geq 0 \quad (53)$$

by axiom 1 $v_i(p)=\beta_i\geq 0$, $\forall p\in[0,1]$ and $\forall i=1,2,...,m$, $\Delta W\left(F\left(x\mid y\right)\right)\geq 0$ if $\mu_{Ai}-\mu_{Bi}\geq 0$, $\forall i=1,...,m$.

For the necessity, suppose that

$$\Delta W\left(F\left(x\mid y\right)\right)=\sum_{i=1}^{m}\beta_{i}\left[\mu_{Ai}-\mu_{Bi}\right]\geq 0$$

but $\exists k=1,...,m$ such that $\mu_{Ak}<\mu_{Bk}$. We can choose a set of numbers $\{\beta_i\}_{i=1,...,m}$ such that $\beta_i\searrow 0$, $\forall i\neq k$. $\Delta W\left(F\left(x\mid y\right)\right)$ would reduce to $\beta_k\left(\mu_{Ak}-\mu_{Bk}\right)<0$, a contradiction. **QED**

Proof of Proposition 8

We want to find a necessary and sufficient condition for

$$\Delta W\left(F\left(x\mid y\right)\right)=\sum_{i=1}^{m}\int_{0}^{1}v_{i}\left(p\right)\left[F_{Ai}^{-1}\left(p\right)-F_{Bi}^{-1}\left(p\right)\right]dp\geq 0,\ \forall W\in\mathbf{W}_{127} \quad (54)$$

For both conditions, note that by axiom 7 we can write: $\Delta W\left(F\left(x\mid y\right)\right)=\sum_{i=1}^{m}\beta_{i}\left[\mu_{Ai}-\mu_{Bi}\right]\geq 0$. Let $S_i=[\mu_{Ai}-\mu_{Bi}]$, $\forall i=1,...,m$. Since by axiom 2 $\beta_i\geq\beta_{i+1}$ and $\beta_i\geq 0$ by axiom 1, we can apply Lemma 1. Therefore, $\Delta W\left(F\left(x\mid y\right)\right)=\sum_{i=1}^{m}\beta_{i}S_{i}\geq 0$ if and only if $\sum_{i=1}^{k}S_i\geq 0$, $\forall k=1,...,m$. Hence, $\Delta W\left(F\left(x\mid y\right)\right)\geq 0$ if and only if $\sum_{i=1}^{k}\mu_{Ai}-\mu_{Bi}$, $\forall k=1,...,m$. **QED**

Proof of Proposition 9

We want to find a necessary and sufficient condition for

$$\Delta W\left(F\left(x \mid y\right)\right)=\sum_{i=1}^{m}\int_{0}^{1}v_{i}\left(p\right)\left[F_{Ai}^{-1}\left(p\right)-F_{Bi}^{-1}\left(p\right)\right]dp\geq 0,\ \forall W\in\mathbf{W}_{1237} \quad (55)$$

For both conditions note that we can apply axiom 1, 2, 3 and 4 and Abel's decomposition as follows: $\Delta W=\sum_{i=1}^{m}\left(\beta_{i}-\beta_{i+1}\right)\sum_{k=1}^{i}S_{k}$, where $S_{k}=\mu_{Ak}-\mu_{Bk}$. Let $\sum_{k=1}^{i}S_{k}=\kappa_{k}$ and $\beta_{i}-\beta_{i+1}=\omega_{i}$, by axiom 3 $\omega_{i}>\omega_{i+1}$, $\forall i=1,...,m-1$. Applying Lemma 1, $\Delta W\left(F\left(x \mid y\right)\right)=\sum_{i=1}^{m}\omega_{i}\kappa_{i}\geq 0$ if and only if $\sum_{i=1}^{j}\sum_{k=1}^{i}\kappa_{k}\geq 0$, $\forall j=1,...,m$. Substituting in the above expression: $\Delta W\left(F\left(x \mid y\right)\right)\geq 0$ if and only if $\sum_{i=1}^{j}\sum_{k=1}^{i}\mu_{Ak}\geq\sum_{i=1}^{j}\sum_{k=1}^{i}\mu_{Bk}$, $\forall j=1,...,m$. **QED**