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OPTIMAL CONTRACT WITH PRIVATE INFORMATION ON COST EXPECTATION AND VOLATILITY

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Very preliminary and incomplete

Abstract

We consider a procurement contract between a principal and an agent who is privately informed about both the expectation and the volatility of the production cost. After the contract is drawn up, the agent privately observes the realized cost as well. We evidence that, in this setting, the principal faces a multi-dimensional screening problem in which relevant decision variables (that affect the agent's incentives) are the expected production and the expected difference between production levels in different states of the world. We characterize the optimal contract and show that its shape depends critically on how the marginal surplus compares with the expected spread between high and low cost realizations.

Keywords: Adverse selection; Multi-dimensional screening; Expectation; Volatility; Marginal surplus

J.E.L. Classification Numbers: D82

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1 Introduction

Most of the studies on principal-agent relationships assume that the agent holds private information about some characteristics (e.g. cost, demand, both cost and demand) that affects the value of the project he is required to develop for the principal.[1] They further assume that, when parties meet at the contracting table, the agent's information about the value of the project is perfect.

Actually, in real world, a large variety of projects have an uncertain return. Take, for instance, a concession contract between the government and a private firm for operating a highway in return for a toll fixed by the government. Typically, at the contracting stage, the net return from operating the highway is uncertain to all parties. Under these circumstances, the information advantage of the firm rests on the latter's ability to form more reliable estimates of the value of the project. The government will thus need to construct an appropriate contractual offer that induces the firm to reveal its true estimate.

Despite situations of this kind are widespread in reality, the literature has devoted little attention to the design of optimal contracts for the accomplishment of projects that have an uncertain outcome in situations where the agent is prively informed about the parameters of the outcome distribution (namely, expected value and volatility).

The aim of this study is to characterize the optimal contract for a situation in which the agent is required to perform some production for the principal and has private information about the expected cost of production and its volatility. This is a two-stage relationship. At the first stage, the contract is signed under uncertainty and private information about the cost distribution parameters. At the second stage, the agent observes the realized cost so that uncertainty is resolved and production takes place.

Our analysis is related to the narrow literature about multidimensional screening mechanisms. Armstrong and Rochet [2] consider a setting in which the agent exerts two activities and holds private information about the technologies that are used to accomplish these two tasks. The principal sets the level of production for the two activities so that the agent has no incentive to misreport on any of the two dimensions. Unlike Armstrong and Rochet [2], we take the agent to undertake one sole activity, with which one sole production level is associated. Nevertheless, our results reveal that, because of the presence of uncertainty, the agent's incentives depend on two measures, namely the expected production and the expected difference between the

[1] See the classical work of Baron and Myerson [3] for the case of private information about the technology.

high and the low production level that the principal commands in different states of the world. Therefore, as in Armstrong and Rochet [2], the principal disposes of two instruments to provide good incentives to the agent. However, unlike in their setting, the two instruments are tied together by the circumstance that they relate to a unique production activity. Because of this, despite a larger number of instruments is at hand, more restrictions appear in terms of contract design. In particular, the principal needs to concede higher rents to induce information revelation. That is, agency costs are more important in the framework we consider.

A multidimensional screening problem is analyzed also in Armstrong [1]. The latter characterizes the optimal contract with private information on demand and cost functions. His model is close to ours in that the agent possesses two pieces of private information but performs one single activity. However, unlike in our framework, the principal relies upon one sole instrument, namely the price, to screen types. This restriction follows from the circumstance that in Armstrong [1] there is no uncertainty in the accomplishment of the task, whereas we refer to situations in which, at the contracting stage, even the agent does not know whether a high or low cost will realize.

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2 The model

We consider a procurement contract between a principal (P) and an agent for the production of q units of some good in turn of a payment t. The expected marginal cost of production θ is drawn from the set $\{\theta_L, \theta_H\}$ with commonly known probabilities ν and $1-\nu$. We denote $\Delta\theta = \theta_H - \theta_L > 0$. The true cost realizes after the contract is signed and before production takes place. It can be either $\theta + \sigma$ or $\theta - \sigma$ with equal probabilities.[2] The uncertain part of the cost σ is drawn from the set $\{\sigma_L, \sigma_H\}$ with commonly known probabilities μ and $1-\mu$. We also denote $\Delta\sigma = \sigma_H - \sigma_L > 0$. We hereafter refer to the generic realization of the two cost parameters as to θ_i and σ_j, with $i, j \in \{L, H\}$. We assume that θ_L, θ_H, σ_L and σ_H are such that $\Delta\theta > \Delta\sigma$ i.e., we require that the spread between the two possible expected unit costs is more important than that between the two possible values of the uncertainty parameter. This is reasonable in that more weight is put on the expected value of the unit production cost than on its volatility.

[2] By attaching equal probabilities to the two possible events, we prevent that the otherwise asymmetric distribution of high and low marginal costs impose structure on the optimal contract.

Information structure When the contract is drawn up, the agent enjoys an information advantage in terms of cost expectation θ_i, $i \in \{L, H\}$, and cost volatility σ_j, $j \in \{L, H\}$, both of which he observes privately before sitting at the contracting table. We denote ij the agent's type for any realized pair (θ_i, σ_j) and $\Upsilon \equiv \{LL, LH, HL, HH\}$ the set of all feasible types. The agent still has an information advantage when the state of nature realizes as he also observes privately whether the latter is $\theta_i + \sigma_j$ or $\theta_i - \sigma_j$. In the sequel, we refer to these two stages of information asymmetry as to the "first" and "second" period.

Payoffs under symmetric information Let $(\underline{q}_{ij}, \underline{t}_{ij})$ and $(\overline{q}_{ij}, \overline{t}_{ij})$ the allocation to be implemented respectively in state $\theta_i - \sigma_j$ and $\theta_i + \sigma_j$, for all $ij \in \Upsilon$. In the two states of nature, the agent's *ex post* profit is respectively given by

$$\overline{\pi}_{ij} = \overline{t}_{ij} - (\theta_i + \sigma_j)\,\overline{q}_{ij} \tag{1}$$

$$\underline{\pi}_{ij} = \underline{t}_{ij} - (\theta_i - \sigma_j)\,\underline{q}_{ij}. \tag{2}$$

In the first period, his payoff is written

$$\Pi_{ij} = \frac{1}{2}(\overline{\pi}_{ij} + \underline{\pi}_{ij}). \tag{3}$$

Production of q units of the good by the agent yields to P gross surplus $S(q)$, which is taken to be three times differentiable to ease the analysis. Specifically, $S(q)$ is such that $S' > 0$, $S'' < 0$ and S''' is finite. The first-period payoff of P is given by

$$V_{ij} = \frac{1}{2}\left[(S(\overline{q}_{ij}) - \overline{t}_{ij}) + (S(\underline{q}_{ij}) - \underline{t}_{ij})\right]. \tag{4}$$

The programme of the principal The Revelation Principle applies. P can restrict attention to truthful direct revelation mechanisms, in which either of the allocations $(\underline{q}_{ij}, \underline{t}_{ij})$ and $(\overline{q}_{ij}, \overline{t}_{ij})$ is implemented depending on the realized state of nature. These allocations are chosen by solving the following programme denoted

Γ :

$$
\begin{aligned}
\underset{(\underline{q}_{ij},\underline{t}_{ij});(\overline{q}_{ij},\overline{t}_{ij})}{Max} W &\equiv \sum_{i=L,H}\sum_{j=L,H}\frac{1}{2}E_{ij}\left[(S(\overline{q}_{ij})-\overline{t}_{ij})+(S(\underline{q}_{ij})-\underline{t}_{ij})\right] \\
& \text{subject to} \\
\Pi_{ij} &\geq \frac{1}{2}\left\{\left[\overline{t}_{i'j'}-(\theta_i+\sigma_j)\,\overline{q}_{i'j'}\right]+\left[\underline{t}_{i'j'}-(\theta_i-\sigma_j)\,\underline{q}_{i'j'}\right]\right\},\ \forall ij;i'j'\in\Upsilon \quad (5) \\
\underline{\pi}_{ij} &\geq \overline{\pi}_{ij}+2\sigma_j\overline{q}_{ij}, ij\in\Upsilon \quad (6) \\
\overline{\pi}_{ij} &\geq \underline{\pi}_{ij}-2\sigma_j\underline{q}_{ij}, ij\in\Upsilon \quad (7) \\
\Pi_{ij} &\geq 0, ij\in\Upsilon. \quad (8)
\end{aligned}
$$

Condition (5) represents the first-period incentive constraints, (6) and (7) the second-period incentive constraints in the two states of nature and (8) the participation constraints for all possible types.

Timing To sum up, the game between P and the agent unfolds as follows. The agent observes privately θ_i and σ_j. P offers to the agent the menu of contracts $\left\{(\underline{q}_{ij},\underline{t}_{ij});(\overline{q}_{ij},\overline{t}_{ij})\right\}$, $\forall ij\in\Upsilon$. The agent reports ij to P and the contract targeted to type ij is signed. The agent observes privately either $\theta_i-\sigma_j$ or $\theta_i+\sigma_j$ and reports the state to P. Accordingly, either the allocation $(\underline{q}_{ij},\underline{t}_{ij})$ or the allocation $(\overline{q}_{ij},\overline{t}_{ij})$ is effected.

2.1 Incentive-compatibility conditions

The second-period incentive constraints (6) and (7) require that $\underline{q}_{ij}\geq\overline{q}_{ij}$ i.e., in the good state $(\theta_i-\sigma_j)$ a larger quantity is to be produced than in the bad state $(\theta_i+\sigma_j)$.

Before looking at the first-period incentive constraints, it is useful to write the expected total cost of the $ij-$agent as

$$EC=\theta_i q_{ij}-\sigma_j r_{ij},$$

where

$$q_{ij}\equiv\frac{1}{2}(\underline{q}_{ij}+\overline{q}_{ij}) \quad \text{and} \quad r_{ij}\equiv\frac{1}{2}(\underline{q}_{ij}-\overline{q}_{ij})$$

respectively denote the expected sum and the expected difference of quantities in the good and bad states. It is straightforward to see that EC increases with the expected production q_{ij} and decreases with the expected production difference r_{ij}, which helps understand how incentives should be given to the agent.[3]

[3]Having the expected cost separable between the two dimensions of asymmetric information

From the first-period incentive constraints we deduce, as usual, a set of conditions on q_{ij} and r_{ij}. First, to prevent the agent from misreporting on one sole dimension, whether i or j, production should be chosen such that

$$q_{Lj} \geq q_{Hj} \ \forall j \in \{L, H\} \quad \text{and} \quad r_{iH} \geq r_{iL} \ \forall i \in \{L, H\}. \tag{9}$$

That is, both the expected production and the expected production difference should be higher for more efficient types. Second, to prevent the agent from cheating simultaneously on both information pieces (θ_i and σ_j), the following additional conditions should be satisfied:

$$\Delta\theta q_{LH} + \Delta\sigma r_{LH} \geq \Delta\theta q_{HL} + \Delta\sigma r_{HL} \tag{10}$$

$$\Delta\theta q_{LL} + \Delta\sigma r_{HH} \geq \Delta\theta q_{HH} + \Delta\sigma r_{LL}. \tag{11}$$

Observe that different rankings of q'_{ij}s and r'_{ij}s, all satisfying (9), (10) and (11), are compatible with the agent's incentive constraints. Albeit it is not immediately apparent which incentive constraints are binding in Γ, it is possible to assess that any incentive-compatible contract is to be structured such that the following lemma holds.

Lemma 1 *At the solution to* Γ, *the conditions*

$$\begin{aligned} \min\{q_{LH}; q_{LL}\} &\geq \max\{q_{HH}; q_{HL}\} \\ \min\{r_{LH}; r_{HH}\} &\geq \max\{r_{LL}; r_{HL}\} \end{aligned} \tag{12}$$

are all necessary, except any of the following: $q_{LL} \geq q_{HH}$, $q_{LH} \geq q_{HL}$, $r_{HH} \geq r_{LL}$, $r_{LH} \geq r_{HL}$.

Proof. See Appendix A. ■

The lemma evidences for which types expected production and expected production differences cannot be unambiguously ranked based on the agent's incentive constraints. Any solution of Γ must be such that q_{ij} and r_{ij} are ranked in a particular order among those in (12).

3 The first-best benchmark

At the first-best outcome (FB hereafter), quantities are such that $S'(\underline{q}^*_{ij}) = \theta_i - \sigma_j$ and $S'(\overline{q}^*_{ij}) = \theta_i + \sigma_j$, whereas $\Pi^*_{ij} = 0$, $\forall ij \in \Upsilon$, the star being appended to

enables us to avoid complications with participation constraints, which arise in non-separable contexts (see Armstrong, 1999).

indicate FB values. Importantly, the specific ranking of q'_{ij}s and r'_{ij}s that arises at the FB solution depends on the shape of P's marginal surplus. In particular, we have $q_{LH} \geq q_{LL}$, $q_{HH} \geq q_{HL}$, $r_{LH} \geq r_{HH}$ and $r_{LL} \geq r_{HL}$ if and only if the marginal surplus declines with quantity at a decreasing rate (i.e. S' is convex). These inequalities are all reversed, instead, when the marginal surplus declines with quantity at an increasing rate (i.e. S' is concave). In the sequel of the analysis, it will become apparent that the shape of S' is also crucial when parties are asymmetrically informed, in which case it dictates the optimal distortions and thus the rents to be given to the various agent's types.

4 The optimal contract

To characterize the optimal contract we need to pin down the solution to Γ. In multi-dimensional problems, the usual approach to pin down the solution is to make a reasonable guess at the relevant incentive constraints" and then check that the solution obtained for this "relaxed" problem satisfies the whole set of constraints. If so, then this is also the solution to the general problem (compare Armstrong [1] and Armstrong and Rochet [2], for instance). This is the strategy we follow in the sequel of the analysis.

4.1 Preliminary

If in our framework it were possible to exactly replicate the "standard" relaxed problem described by Armstrong and Rochet [2], then our best guess at relevant constraints would be that binding are the participation constraint of the least efficient type, the incentive constraints whereby neither of the intermediate types LL and HH being tempted to mimic type HL,[4] the tightest incentive constraint of the most efficient type LH. This would yield the following list of rents:

$$\Pi_{HL} = 0 \quad (13)$$

$$\Pi_{HH} = \Delta\sigma r_{HL} \quad (14)$$

$$\Pi_{LL} = \Delta\theta q_{HL} \quad (15)$$

$$\Pi_{LH} = \max\{\Delta\theta q_{HH} + \Delta\sigma r_{HL};\ \Delta\theta q_{HL} + \Delta\sigma r_{LL};\ \Delta\theta q_{HL} + \Delta\sigma r_{HL}\} \quad (16)$$

Let us denote this relaxed problem Γ_{AR}, which stays for relaxed problem "*à la* Armstrong and Rochet [2]". If the solution to Γ_{AR} were to satisfy all the other constraints as well, then it would be also the solution to Γ, provided the rents

[4] By "intermediate" types we mean types displaying an intermediate degree of cost efficiency.

above are least costly to the principal (hence, they are the "ideal" ones). The lemma hereafter states that the guess of Armstrong and Rochet [2] does not suit the situation we consider.

Lemma 2 *The solution to* Γ_{AR} *is not the solution to* Γ.

Proof. See Appendix B.1. ■

Although the details of the proof of this result are reported in the Appendix, it is worth suggesting that, to understand it fully, one should focus on the intermediate types LL and HH. If P assigns the rents (13) to (16), then the incentive constraint that prevents type LL from mimicking type HH is written

$$\Delta\sigma \left(r_{HH} - r_{HL}\right) \geq \Delta\theta \left(q_{HH} - q_{HL}\right)$$

or, equivalently,

$$\left(\Delta\theta - \Delta\sigma\right)\left(\underline{q}_{HL} - \underline{q}_{HH}\right) + \left(\Delta\theta + \Delta\sigma\right)\left(\overline{q}_{HL} - \overline{q}_{HH}\right) \geq 0$$

where we have used the definitions of q_{ij} and r_{ij}. The difference $\left(\underline{q}_{HL} - \underline{q}_{HH}\right)$ is negative (i.e. $\underline{q}_{HH} > \underline{q}_{HL}$) both because type HH is more efficient in the good state, so that the first-best quantities are ranked as $\underline{q}^*_{HH} > \underline{q}^*_{HL}$, and because P distorts the $HL-$quantity downwards so as to contain the information rents.[5] Therefore, for the incentive constraint to be satisfied, it must be the case that the difference $\left(\overline{q}_{HL} - \overline{q}_{HH}\right)$ is positive. This involves that the quantities assigned to type HL in the two states of nature are more dispersed than those assigned to type HH. It follows that $r_{HL} > r_{HH}$. However, if production is set so as to satisfy the latter inequality, then it is not possible to prevent type HL from selecting the $HH-$contract without conceding an information rent to this type.

It is thus clear that the initial guess at relevant incentive constraints was not correct. We hereafter present the guess that is appropriate to our framework.

4.2 The relaxed problem

We have seen that in Γ_{AR} both intermediate types (HH and LL) are tempted to mimic the least efficient type HL. We shall now consider another relaxed problem in which either such type has an incentive to mimic the type that is immediately below in the efficiency ranking. That is, type HH would like to mimic HL and type LL would like to mimic HH. We denote this problem Γ' so as to avoid confusion with the relaxed problem "*à la* Armstrong and Rochet [2]" that we have previously

[5] It is immediate to see that rents increase with both q_{HL} and r_{HL}, hence with $\underline{q}_{HL}$.

discussed. In Γ', the incentive constraint whereby type LL not being tempted to mimic type HH is binding, that is

$$\Pi_{LL} = \Pi_{HH} + \Delta\theta q_{HH} - \Delta\sigma r_{HH},$$

whereas type LL has no interest in pretending to be type HL. Under these circumstances, the rents are given by (13) and (14) together with

$$\Pi_{LL} = \Delta\theta q_{HH} + \Delta\sigma r_{HL} - \Delta\sigma r_{HH} \tag{17}$$

$$\Pi_{LH} = \max\{\Delta\theta q_{HH} + \Delta\sigma r_{HL};\ . \tag{18}$$
$$\Delta\theta q_{HH} + \Delta\sigma r_{HL} + \Delta\sigma r_{LL} - \Delta\sigma r_{HH}; \Delta\theta q_{HL} + \Delta\sigma r_{HL}\}\ .$$

Specifically, the rent in (18) means that we allow type LH to be possibly attracted by the contract of any other type.

Actually, the possibility that $\Pi_{LH} = \Delta\theta q_{HL} + \Delta\sigma r_{HL}$, which occurs whenever the most efficient type LH has an incentive to mimic the least efficient type HL, can be immediately ruled out. The following lemma states this result.

Lemma 3 *There exists no solution to* Γ *such that* P *assigns the rents (13), (14), (17) and* $\Pi_{LH} = \Delta\theta q_{HL} + \Delta\sigma r_{HL}$.

Proof. See Appendix B.2. ■

The interpretation of this result requires some caution. While Lemma 3 does mean that type LH never has a strict preference to mimic type HL over any other type, it does not mean that type LH cannot be indifferent between mimicking type HL and any other type. That is, the rent $\Delta\theta q_{HL} + \Delta\sigma r_{HL}$ can still be assigned to the most efficient type if the gain from pretending to be type HL is the same as that from pretending to be any of the other two types.

TO BE COMPLETED

References

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A Proof of Lemma 1

After listing all the incentive compatibility constraints, we develop the following steps. First, we assume that the inequality between some q_{ij} and $q_{i'j'}$ is satisfied. Subsequently, using the incentive constraints, we find which other conditions this inequality implies. We proceed similarly with r_{ij} and repeat the procedure for all types. We obtain a number of bunches of conditions. At least one of them must be satisfied at the solution. We compare all bunches of conditions and identify those that are implied by others. From the remaining relevant cases, we derive the conditions reported in the lemma.

A.1 Incentive compatibility constraints

Here is the full list of incentive compatibility constraints:

$$\Pi_{LL} \geq \Pi_{HL} + \Delta\theta q_{HL} \tag{1}$$
$$\Pi_{LL} \geq \Pi_{LH} - \Delta\sigma r_{LH} \tag{2}$$
$$\Pi_{LL} \geq \Pi_{HH} + \Delta\theta q_{HH} - \Delta\sigma r_{HH} \tag{3}$$
$$\Pi_{HL} \geq \Pi_{LL} - \Delta\theta q_{LL} \tag{4}$$
$$\Pi_{HL} \geq \Pi_{HH} - \Delta\sigma r_{HH} \tag{5}$$
$$\Pi_{HL} \geq \Pi_{LH} - \Delta\theta q_{LH} - \Delta\sigma r_{LH} \tag{6}$$
$$\Pi_{LH} \geq \Pi_{HH} + \Delta\theta q_{HH} \tag{7}$$
$$\Pi_{LH} \geq \Pi_{LL} + \Delta\sigma r_{LL} \tag{8}$$
$$\Pi_{LH} \geq \Pi_{HL} + \Delta\theta q_{HL} + \Delta\sigma r_{HL} \tag{9}$$
$$\Pi_{HH} \geq \Pi_{LH} - \Delta\theta q_{LH} \tag{10}$$
$$\Pi_{HH} \geq \Pi_{HL} + \Delta\sigma r_{HL} \tag{11}$$
$$\Pi_{HH} \geq \Pi_{LL} - \Delta\theta q_{LL} + \Delta\sigma r_{LL} \tag{12}$$

A.2 Feasible cases for ranking q_{ij} and r_{ij}

1) Assume $q_{HL} \geq q_{LH}$. Jointly with (10), this implies $r_{LH} \geq r_{HL}$. Using (9), we then find

$$\begin{cases} q_{LL} \geq q_{HL} \geq q_{LH} \geq q_{HH} \\ r_{HH} \geq r_{HL};\ r_{LH} \geq r_{HL};\ r_{LH} \geq r_{LL}. \end{cases} \tag{19}$$

2) Assume $r_{HL} \geq r_{LH}$. Jointly with (10), this implies $q_{LH} \geq q_{HL}$, so that overall we have

$$\begin{cases} q_{LH} \geq q_{HH};\ q_{LH} \geq q_{HL};\ q_{LL} \geq q_{HL} \\ r_{HH} \geq r_{HL} \geq r_{LH} \geq r_{LL}. \end{cases} \tag{20}$$

3) Assume $q_{HH} \geq q_{LL}$. Jointly with (11), this implies $r_{HH} \geq r_{LL}$ and so

$$\begin{cases} q_{LH} \geq q_{HH} \geq q_{LL} \geq q_{HL} \\ r_{HH} \geq r_{HL};\ r_{HH} \geq r_{LL};\ r_{LH} \geq r_{LL}. \end{cases} \tag{21}$$

4) Assume $r_{LL} \geq r_{HH}$. Jointly with (11), this implies $q_{LL} \geq q_{HH}$ and so

$$\left\{ \begin{array}{c} q_{LH} \geq q_{HH};\ q_{LL} \geq q_{HH};\ q_{LL} \geq q_{HL} \\ r_{LH} \geq r_{LL} \geq r_{HH} \geq r_{HL}. \end{array} \right. \tag{22}$$

The remaining cases are the converse of any of the situations 1) - 4) above. We look at them hereafter.

5) Take the converse of 1), that is $q_{LH} \geq q_{HL}$. Then, either $r_{LH} \geq r_{HL}$ or $r_{HL} \geq r_{LH}$. The latter case is identical to 2). Suppose $r_{LH} \geq r_{HL}$. If also $q_{HH} \geq q_{LL}$, then we are back to 3). Suppose $q_{LL} \geq q_{HH}$. Then, we get

$$\left\{ \begin{array}{c} q_{LH} \geq q_{HL};\ q_{LH} \geq q_{HH};\ q_{LL} \geq q_{HL};\ q_{LL} \geq q_{HH} \\ r_{LH} \geq r_{HL};\ r_{LH} \geq r_{LL};\ r_{HH} \geq r_{HL}. \end{array} \right. \tag{23}$$

Still taking $q_{LH} \geq q_{HL}$ and $r_{LH} \geq r_{HL}$ and further assuming $r_{LL} \geq r_{HH}$, we are back to 4). Next suppose $r_{HH} \geq r_{LL}$. Then, we have

$$\left\{ \begin{array}{c} q_{LH} \geq q_{HL};\ q_{LH} \geq q_{HH};\ q_{LL} \geq q_{HL} \\ r_{LH} \geq r_{HL};\ r_{LH} \geq r_{LL};\ r_{HH} \geq r_{HL};\ r_{HH} \geq r_{LL}. \end{array} \right. \tag{24}$$

6) Assume the converse of 2), that is $r_{LH} \geq r_{HL}$. If $q_{HL} \geq q_{LH}$, then we are back to 1). If $q_{LH} \geq q_{HL}$, then we are back to 5).

7) Assume the converse of 3), that is $q_{LL} \geq q_{HH}$. If $r_{LL} \geq r_{HH}$, then we find again 4). Suppose $r_{HH} \geq r_{LL}$. If $q_{HL} \geq q_{LH}$, then we get 1). Supposing $q_{LH} \geq q_{HL}$, we obtain

$$\left\{ \begin{array}{c} q_{LH} \geq q_{HL};\ q_{LL} \geq q_{HH};\ q_{LL} \geq q_{HL};\ q_{LH} \geq q_{HH} \\ r_{HH} \geq r_{LL};\ r_{HH} \geq r_{HL};\ r_{LH} \geq r_{LL}. \end{array} \right. \tag{25}$$

Still taking $q_{LL} \geq q_{HH}$ and $r_{HH} \geq r_{LL}$ and further assuming $r_{HL} \geq r_{LH}$, we move back to 2). Next suppose $r_{LH} \geq r_{HL}$. We have

$$\left\{ \begin{array}{c} q_{LL} \geq q_{HH};\ q_{LL} \geq q_{HL};\ q_{LH} \geq q_{HH} \\ r_{HH} \geq r_{LL};\ r_{HH} \geq r_{HL};\ r_{LH} \geq r_{LL};\ r_{LH} \geq r_{HL}. \end{array} \right. \tag{26}$$

8) Assume the converse of 4), that is $r_{HH} \geq r_{LL}$. If $q_{HH} \geq q_{LL}$, then we are back to 3). Supposing $q_{LL} \geq q_{HH}$, we come back to (26) in the second part of 7).

A.3 Remove irrelevant cases

The first line in (19) is a particular case of first line in (22). Moreover, the second line in (22) is a particular case of second line in (19). Overall, (19) and (22) must hold simultaneously. We obtain the following ranking:

$$\begin{cases} q_{LL} \geq q_{HL} \geq q_{LH} \geq q_{HH} \\ r_{LH} \geq r_{LL} \geq r_{HH} \geq r_{HL}. \end{cases} \tag{27}$$

Under (27) the only relevant incentive constraints are (1), (5), (8) and (10) as the others are all slack. These constraints cannot hold simultaneously unless both $q_{HL} = q_{LH}$ and $r_{LL} = r_{HH}$. Replacing into (27) we deduce that (27) is a particular case of

$$\begin{cases} q_{LL} \geq q_{LH} \geq q_{HL} \geq q_{HH} \\ r_{LH} \geq r_{HH} \geq r_{LL} \geq r_{HL}, \end{cases}$$

which is in turn a particular situation of all cases (23) to (26). Hence, (19) and (22) can be omitted. Following the same reasoning as for (19) and (22), we find that (20) and (21) must hold simultaneously. The ranking becomes

$$\begin{aligned} q_{LH} &\geq q_{HH} \geq q_{LL} \geq q_{HL} \\ r_{HH} &\geq r_{HL} \geq r_{LH} \geq r_{LL}. \end{aligned} \tag{28}$$

Moreover, under (28) the only relevant constraints are (4), (2), (7) and (11) as the others are all slack. These constraints cannot hold simultaneously unless both $q_{HH} = q_{LL}$ and $r_{HL} = r_{LH}$. Replacing into (28) we find that (28) is a particular case of

$$\begin{aligned} q_{LH} &\geq q_{LL} \geq q_{HH} \geq q_{HL} \\ r_{HH} &\geq r_{LH} \geq r_{HL} \geq r_{LL}, \end{aligned}$$

which is in turn another particular situation of all cases (23) to (26). Hence, (20) and (21) can be omitted too. Putting together the remaining conditions (23) to (26), we obtain the conditions in the lemma.

B The optimal contract

B.1 Proof of Lemma 2

The relaxed problem with rents (13) to (16), denoted Γ_{AR}, is such that these rents replace (5) in Γ. To assess how the rent specifies in (16), we follow Armstrong and Rochet [2] and attach the multipliers $\gamma_1, \gamma_2, \gamma_3 \in [0,1]$, such that $\gamma_1 + \gamma_2 + \gamma_3 = 1$, to the three downward incentive constraints of type LH and rewrite (16) as

$$\Pi_{LH} = \gamma_1 (\Delta\theta q_{HH} + \Delta\sigma r_{HL}) + \gamma_2 (\Delta\theta q_{HL} + \Delta\sigma r_{LL}) + \gamma_3 (\Delta\theta q_{HL} + \Delta\sigma r_{HL}).$$

If any γ_i, $i \in \{1,2,3\}$, takes a positive value, then the associated incentive constraint is binding and the corresponding rent is largest.

The expected rent of the agent is written

$$\nu\mu\Delta\theta q_{HL} - (1-\nu)(1-\mu)\Delta\sigma r_{HL} + \nu(1-\mu)[\gamma_1(\Delta\theta q_{HH} + \Delta\sigma r_{HL}) \\ +\gamma_2(\Delta\theta q_{HL} + \Delta\sigma r_{LL}) + \gamma_3(\Delta\theta q_{HL} + \Delta\sigma r_{HL})],$$

so that P's expected utility is given by

$$\begin{aligned} W &= \frac{1}{2}\nu\mu\left[S(\underline{q}_{LL}) - (\theta_L - \sigma_L) + S(\overline{q}_{LL}) - (\theta_L + \sigma_L)\right] \\ &\quad + \frac{1}{2}\nu(1-\mu)\left[S(\underline{q}_{LH}) - (\theta_L - \sigma_H) + S(\overline{q}_{LH}) - (\theta_L + \sigma_H)\right] \\ &\quad + \frac{1}{2}(1-\nu)\mu\left[S(\underline{q}_{HL}) - (\theta_H - \sigma_L) + S(\overline{q}_{HL}) - (\theta_H + \sigma_L)\right] \\ &\quad + \frac{1}{2}(1-\nu)(1-\mu)\left[S(\underline{q}_{HH}) - (\theta_H - \sigma_H) + S(\overline{q}_{HH}) - (\theta_H + \sigma_H)\right] \\ &\quad - \nu\mu\Delta\theta q_{HL} - (1-\nu)(1-\mu)\Delta\sigma r_{HL} + \nu(1-\mu)[\gamma_1(\Delta\theta q_{HH} + \Delta\sigma r_{HL}) \\ &\quad + \gamma_2(\Delta\theta q_{HL} + \Delta\sigma r_{LL}) + \gamma_3(\Delta\theta q_{HL} + \Delta\sigma r_{HL})]. \end{aligned}$$

The resulting quantity solution is characterized as follows. For type LH, production is set at the FB level in either state of the world:

$$S'(\underline{q}^*_{LH}) = \theta_L - \sigma_H \tag{29}$$
$$S'(\overline{q}^*_{LH}) = \theta_L + \sigma_H. \tag{30}$$

For type LL, production is downward distorted in the good state and upward distorted in the bad state:

$$S'(\underline{q}_{LL}) = \theta_L - \sigma_L + \gamma_2\frac{1-\mu}{\mu}\Delta\sigma \tag{31}$$
$$S'(\overline{q}_{LL}) = \theta_L + \sigma_L - \gamma_2\frac{1-\mu}{\mu}\Delta\sigma. \tag{32}$$

For type HH, production is downward distorted in either state:

$$S'(\underline{q}_{HH}) = \theta_H - \sigma_H + \gamma_1\frac{\nu}{1-\nu}\Delta\theta \tag{33}$$
$$S'(\overline{q}_{HH}) = \theta_H + \sigma_H + \gamma_1\frac{\nu}{1-\nu}\Delta\theta. \tag{34}$$

Lastly, for type HL, production is downward distorted in the good state and can be either downward or upward distorted in the bad state:

$$S'(\underline{q}_{HL}) = \theta_H - \sigma_L + \frac{\nu}{1-\nu}\left[1 + \frac{1-\mu}{\mu}(\gamma_2+\gamma_3)\right]\Delta\theta \tag{35}$$
$$+\frac{1-\mu}{\mu}\left[1+\frac{\nu}{1-\nu}(\gamma_1+\gamma_3)\right]\Delta\sigma$$
$$S'(\overline{q}_{HL}) = \theta_H + \sigma_L + \frac{\nu}{1-\nu}\left[1 + \frac{1-\mu}{\mu}(\gamma_2+\gamma_3)\right]\Delta\theta \tag{36}$$
$$-\frac{1-\mu}{\mu}\left[1+\frac{\nu}{1-\nu}(\gamma_1+\gamma_3)\right]\Delta\sigma.$$

It is easy to check that this solution violates incentive constraint (3), as suggested in the main text.

B.2 Proof of Lemma 3

It is straightforward to see that, with rents (13), (14), (17) and (18), all expected profits are non-negative. Therefore, none of these rents violates the agent's participation constraints.

Let us next see whether and under which conditions the incentive constraints are satisfied when those same rents are assigned. Replacing them we have

$$\Delta\theta q_{HH} + \Delta\sigma r_{HL} \geq \Delta\theta q_{HL} + \Delta\sigma r_{HH} \tag{1}$$
$$\Delta\theta q_{HH} + \Delta\sigma r_{LH} \geq \Delta\theta q_{HL} + \Delta\sigma r_{HH} \tag{2}$$
$$\Delta\theta q_{LL} + \Delta\sigma r_{HH} \geq \Delta\theta q_{HH} + \Delta\sigma r_{HL} \tag{4}$$
$$r_{HH} \geq r_{HL} \tag{5}$$
$$\Delta\theta q_{LH} + \Delta\sigma r_{LH} \geq \Delta\theta q_{HL} + \Delta\sigma r_{HL} \tag{6}$$
$$q_{HL} \geq q_{HH} \tag{7}$$
$$\Delta\theta q_{HL} + \Delta\sigma r_{HL} \geq \Delta\theta q_{HH} + \Delta\sigma r_{LL} - \Delta\sigma r_{HH} \tag{8}$$
$$q_{LH} \geq q_{HL} \tag{10}$$
$$\Delta\theta q_{LL} + \Delta\sigma r_{HH} \geq \Delta\theta q_{HH} + \Delta\sigma r_{LL}. \tag{12}$$

For the incentive constraints to be satisfied, necessary conditions are $q_{LH} \geq q_{HL} \geq q_{HH}$ and $r_{HH} \geq r_{HL}$. Suppose they hold. Then, we remain with

$$\Delta\theta q_{HH} + \Delta\sigma r_{HL} \geq \Delta\theta q_{HL} + \Delta\sigma r_{HH} \tag{1}$$
$$\Delta\theta q_{HH} + \Delta\sigma r_{LH} \geq \Delta\theta q_{HL} + \Delta\sigma r_{HH} \tag{2}$$
$$\Delta\theta q_{LL} + \Delta\sigma r_{HH} \geq \Delta\theta q_{HH} + \Delta\sigma r_{HL} \tag{4}$$
$$\Delta\theta q_{LH} + \Delta\sigma r_{LH} \geq \Delta\theta q_{HL} + \Delta\sigma r_{HL} \tag{6}$$
$$\Delta\theta q_{HL} + \Delta\sigma r_{HL} \geq \Delta\theta q_{HH} + \Delta\sigma r_{LL} - \Delta\sigma r_{HH} \tag{8}$$
$$\Delta\theta q_{LL} + \Delta\sigma r_{HH} \geq \Delta\theta q_{HH} + \Delta\sigma r_{LL}. \tag{12}$$

With $q_{HL} \geq q_{HH}$, (1) would require that $r_{HL} \geq r_{HH}$. This contradicts the necessary condition previously identified.

B.3 The quantity solution to Γ'

Following again the methodology proposed by Armstrong and Rochet [2], we attach the multipliers α_1, α_2, $\alpha_3 \in [0,1]$, such that $\alpha_1 + \alpha_2 + \alpha_3 = 1$, to the three downward incentive constraints of type LH and rewrite (18) as

$$\begin{aligned}\Pi_{LH} &= \alpha_1 (\Delta\theta q_{HH} + \Delta\sigma r_{HL}) + \alpha_2 (\Delta\theta q_{HH} + \Delta\sigma r_{HL} + \Delta\sigma r_{LL} - \Delta\sigma r_{HH}) \\ &\quad + \alpha_3 (\Delta\theta q_{HL} + \Delta\sigma r_{HL}).\end{aligned}$$

If any α_i, $i \in \{1,2,3\}$, takes a positive value, then the associated incentive constraint is binding and the corresponding rent is largest.

The expected rent is written

$$\begin{aligned}&[\nu\mu + \nu(1-\mu)(\alpha_1+\alpha_2)]\Delta\theta q_{HH} - [\nu\mu + \nu(1-\mu)\alpha_2]\Delta\sigma r_{HH} \\ &+\nu(1-\mu)(\alpha_2\Delta\sigma r_{LL} + \alpha_3\Delta\theta q_{HL}) + [1-\mu(1-\nu)]\Delta\sigma r_{HL},\end{aligned}$$

so that the objective function of P is given by

$$\begin{aligned}W &= \frac{1}{2}\nu\mu\left[S(\underline{q}_{LL}) - (\theta_L - \sigma_L) + S(\overline{q}_{LL}) - (\theta_L + \sigma_L)\right] \\ &\quad + \frac{1}{2}\nu(1-\mu)\left[S(\underline{q}_{LH}) - (\theta_L - \sigma_H) + S(\overline{q}_{LH}) - (\theta_L + \sigma_H)\right] \\ &\quad + \frac{1}{2}(1-\nu)\mu\left[S(\underline{q}_{HL}) - (\theta_H - \sigma_L) + S(\overline{q}_{HL}) - (\theta_H + \sigma_L)\right] \\ &\quad + \frac{1}{2}(1-\nu)(1-\mu)\left[S(\underline{q}_{HH}) - (\theta_H - \sigma_H) + S(\overline{q}_{HH}) - (\theta_H + \sigma_H)\right] \\ &\quad - [\nu\mu + \nu(1-\mu)(\alpha_1+\alpha_2)]\Delta\theta q_{HH} + [\nu\mu + \nu(1-\mu)\alpha_2]\Delta\sigma r_{HH} \\ &\quad - \nu(1-\mu)(\alpha_2\Delta\sigma r_{LL} + \alpha_3\Delta\theta q_{HL}) - [1-\mu(1-\nu)]\Delta\sigma r_{HL}.\end{aligned}$$

The resulting quantity solution is characterized as follows. For type LH, production is set at the FB level in either state of the world, i.e. (29) and (30) still hold. For type LL production is again downward distorted in the good state and upward distorted in the bad state:

$$S'(\underline{q}_{LL}) = \theta_L - \sigma_L + \alpha_2 \frac{1-\mu}{\mu}\Delta\sigma \tag{37}$$

$$S'(\overline{q}_{LL}) = \theta_L + \sigma_L - \alpha_2 \frac{1-\mu}{\mu}\Delta\sigma. \tag{38}$$

For type HH production is either downward or upward distorted in the good state and downward distorted in the bad state:

$$\begin{aligned} S'(\underline{q}_{HH}) &= \theta_H - \sigma_H + \frac{\nu\mu + \nu(1-\mu)(\alpha_1+\alpha_2)}{(1-\nu)(1-\mu)}\Delta\theta \\ &\quad - \frac{\nu\mu + \nu(1-\mu)\alpha_2}{(1-\nu)(1-\mu)}\Delta\sigma \end{aligned} \tag{39}$$

$$\begin{aligned} S'(\overline{q}_{HH}) &= \theta_H + \sigma_H + \frac{\nu\mu + \nu(1-\mu)(\alpha_1+\alpha_2)}{(1-\nu)(1-\mu)}\Delta\theta \\ &\quad + \frac{\nu\mu + \nu(1-\mu)\alpha_2}{(1-\nu)(1-\mu)}\Delta\sigma. \end{aligned} \tag{40}$$

For type HL production is downward distorted in the good state and either downward or upward distorted in the bad state:

$$S'(\underline{q}_{HL}) = \theta_H - \sigma_L + \alpha_3\frac{\nu(1-\mu)}{(1-\nu)\mu}\Delta\theta + \frac{1-\mu(1-\nu)}{(1-\nu)\mu}\Delta\sigma \tag{41}$$

$$S'(\overline{q}_{HL}) = \theta_H + \sigma_L + \alpha_3\frac{\nu(1-\mu)}{(1-\nu)\mu}\Delta\theta - \frac{1-\mu(1-\nu)}{(1-\nu)\mu}\Delta\sigma. \tag{42}$$

B.4 The monotonicity conditions in Γ'

Types LH and LL We have

$$\begin{aligned} S'(\underline{q}_{LH}) - S'(\underline{q}_{LL}) &= -\left(1+\alpha_2\frac{1-\mu}{\mu}\right)\Delta\sigma < 0 \\ S'(\overline{q}_{LH}) - S'(\overline{q}_{LL}) &= \left(1+\alpha_2\frac{1-\mu}{\mu}\right)\Delta\sigma > 0, \end{aligned}$$

so that $\underline{q}_{LH} > \underline{q}_{LL}$ and $\overline{q}_{LL} > \overline{q}_{LH}$. Hence, $r_{LH} > r_{LL}$. Moreover,

$$S'(\underline{q}_{LL}) - S'(\underline{q}_{LH}) = S'(\overline{q}_{LH}) - S'(\overline{q}_{LL}),$$

so that $q_{LH} \geq q_{LL}$ if and only if S' is convex.

Types LH and HL We calculate

$$\begin{aligned} S'(\underline{q}_{HL}) - S'(\underline{q}_{LH}) &= \Delta\theta + \Delta\sigma + \alpha_3\frac{\nu(1-\mu)}{(1-\nu)\mu}\Delta\theta + \frac{1-\mu(1-\nu)}{(1-\nu)\mu}\Delta\sigma > 0 \\ S'(\overline{q}_{LH}) - S'(\overline{q}_{HL}) &= -\Delta\theta + \Delta\sigma - \alpha_3\frac{\nu(1-\mu)}{(1-\nu)\mu}\Delta\theta + \frac{1-\mu(1-\nu)}{(1-\nu)\mu}\Delta\sigma. \end{aligned}$$

Hence, $\underline{q}_{LH} > \underline{q}_{HL}$. We also have $\overline{q}_{HL} > \overline{q}_{LH}$ if and only if $[S'(\overline{q}_{LH}) - S'(\overline{q}_{HL})] > 0$ and so if and only if

$$\frac{\Delta\theta}{\Delta\sigma} < \frac{1}{(1-\nu)\mu + \alpha_3\nu(1-\mu)}, \tag{c1}$$

in which case $r_{LH} > r_{HL}$. We also calculate

$$\left[S'(\underline{q}_{HL}) - S'(\underline{q}_{LH})\right] - [S'(\overline{q}_{LH}) - S'(\overline{q}_{HL})] = 2\Delta\theta + 2\alpha_3 \frac{\nu(1-\mu)}{(1-\nu)\mu}\Delta\theta > 0,$$

which tells that $q_{LH} > q_{HL}$ if and only if S' is not too concave. If

$$\frac{\Delta\theta}{\Delta\sigma} > \frac{1}{(1-\nu)\mu + \alpha_3 \nu (1-\mu)},$$

then $\overline{q}_{LH} > \overline{q}_{HL}$ and so $q_{LH} > q_{HL}$. We further compute

$$\left[S'(\underline{q}_{HL}) - S'(\underline{q}_{LH})\right] - [S'(\overline{q}_{HL}) - S'(\overline{q}_{LH})] = 2\Delta\sigma + 2\frac{1-\mu(1-\nu)}{(1-\nu)\mu}\Delta\sigma > 0$$

Hence, $r_{LH} > r_{HL}$ if and only S' is not too concave.

Overall, if (c1) holds, then it is $r_{LH} > r_{HL}$ whereas $q_{LH} > q_{HL}$ if and only if S' is not too concave.

If (c1) does not hols, then $q_{LH} > q_{HL}$ whereas $r_{LH} > r_{HL}$ if and only if S' is not too concave.

Types LH **and** HH We have

$$\begin{aligned} S'(\underline{q}_{LH}) - S'(\underline{q}_{HH}) &= -\left[1 + \frac{\nu\mu + \nu(1-\mu)(\alpha_1+\alpha_2)}{(1-\nu)(1-\mu)}\right]\Delta\theta + \frac{\nu\mu + \nu(1-\mu)\alpha_2}{(1-\nu)(1-\mu)}\Delta\sigma < 0 \\ S'(\overline{q}_{LH}) - S'(\overline{q}_{HH}) &= -\left[1 + \frac{\nu\mu + \nu(1-\mu)(\alpha_1+\alpha_2)}{(1-\nu)(1-\mu)}\right]\Delta\theta - \frac{\nu\mu + \nu(1-\mu)\alpha_2}{(1-\nu)(1-\mu)}\Delta\sigma < 0, \end{aligned}$$

so that both $\underline{q}_{LH} > \underline{q}_{HH}$ and $\overline{q}_{LH} > \overline{q}_{HH}$. Hence, $q_{LH} > q_{HH}$. We also calculate

$$\left[S'(\underline{q}_{HH}) - S'(\underline{q}_{LH})\right] - [S'(\overline{q}_{HH}) - S'(\overline{q}_{LH})] = -2\frac{\nu\mu + \nu(1-\mu)\alpha_2}{(1-\nu)(1-\mu)}\Delta\sigma < 0,$$

so that $r_{LH} > r_{HH}$ if and only if S' is sufficiently convex. In particular, $r_{LH} < r_{HH}$ with S' linear.

Types LL **and** HL Let us compute

$$\begin{aligned} S'(\underline{q}_{HL}) - S'(\underline{q}_{LL}) &= \left[1 + \alpha_3 \frac{\nu(1-\mu)}{(1-\nu)\mu}\right]\Delta\theta + \left[\frac{1-\mu(1-\nu)}{(1-\nu)\mu} - \alpha_2 \frac{1-\mu}{\mu}\right]\Delta\sigma > 0 \\ S'(\overline{q}_{LL}) - S'(\overline{q}_{HL}) &= -\left[1 + \alpha_3 \frac{\nu(1-\mu)}{(1-\nu)\mu}\right]\Delta\theta + \left[\frac{1-\mu(1-\nu)}{(1-\nu)\mu} - \alpha_2 \frac{1-\mu}{\mu}\right]\Delta\sigma \end{aligned}$$

Hence, $\underline{q}_{LL} > \underline{q}_{HL}$. If

$$\frac{\Delta\theta}{\Delta\sigma} < \frac{1-\mu(1-\nu)-\alpha_2(1-\mu)(1-\nu)}{\alpha_3\nu(1-\mu)+(1-\nu)\mu}, \tag{c2}$$

then $[S'(\overline{q}_{LL})-S'(\overline{q}_{HL})] > 0$ and so $\overline{q}_{HL} > \overline{q}_{LL}$. With $\alpha_2 = 1$ the RHS of (c2) becomes $\frac{\nu}{(1-\nu)\mu}$, which can be larger than 1. With $\alpha_3 = 1$ the RHS is $\frac{1-\mu+\nu\mu}{\nu(1-\mu)+(1-\nu)\mu}$, which exceeds 1 if and only if $(1-\nu)(1-2\mu) > 0$. This is feasible. If (c2) satisfied, then we have $r_{LL} > r_{HL}$. We also compute

$$\left[S'(\underline{q}_{HL})-S'(\underline{q}_{LL})\right]-[S'(\overline{q}_{LL})-S'(\overline{q}_{HL})] = 2\left[1+\alpha_3\frac{\nu(1-\mu)}{(1-\nu)\mu}\right]\Delta\theta > 0,$$

so that $q_{LL} > q_{HL}$ if and only if S' is not too concave. If (c2) is not satisfied, then $\overline{q}_{LL} > \overline{q}_{HL}$. In this case $q_{LL} > q_{HL}$. Moreover,

$$\left[S'(\underline{q}_{HL})-S'(\underline{q}_{LL})\right]-[S'(\overline{q}_{HL})-S'(\overline{q}_{LL})] = 2\frac{1-(1-\nu)[\mu(1-\alpha_2)+\alpha_2]}{(1-\nu)\mu}\Delta\sigma > 0,$$

so that $r_{LL} > r_{HL}$ if and only if S' is not too concave.

Overall, if (c2) holds, then $q_{LL} > q_{HL}$ if and only if S' is not too concave together with $r_{LL} > r_{HL}$.

If (c2) does not hold, then $q_{LL} > q_{HL}$ together with $r_{LL} > r_{HL}$ if and only if S' is not too concave.

Types LL and HH We calculate

$$\begin{aligned} S'(\underline{q}_{LL})-S'(\underline{q}_{HH}) &= -\left[1+\frac{\nu\mu+\nu(1-\mu)(\alpha_1+\alpha_2)}{(1-\nu)(1-\mu)}\right]\Delta\theta \\ &\quad +\left[1+\alpha_2\frac{1-\mu}{\mu}+\frac{\nu\mu+\nu(1-\mu)\alpha_2}{(1-\nu)(1-\mu)}\right]\Delta\sigma \\ S'(\overline{q}_{HH})-S'(\overline{q}_{LL}) &= \left[1+\frac{\nu\mu+\nu(1-\mu)(\alpha_1+\alpha_2)}{(1-\nu)(1-\mu)}\right]\Delta\theta \\ &\quad +\left[1+\alpha_2\frac{1-\mu}{\mu}+\frac{\nu\mu+\nu(1-\mu)\alpha_2}{(1-\nu)(1-\mu)}\right]\Delta\sigma \end{aligned}$$

Hence, $\overline{q}_{LL} > \overline{q}_{HH}$. If

$$\frac{\Delta\theta}{\Delta\sigma} < \frac{\nu\mu+(1-\nu)(1-\mu)+\alpha_2(1-\mu)\left[\frac{1-\mu}{\mu}(1-\nu)+\nu\right]}{\nu\mu+(1-\nu)(1-\mu)+\nu(1-\mu)(\alpha_1+\alpha_2)}, \tag{c3}$$

then $\left[S'(\underline{q}_{LL})-S'(\underline{q}_{HH})\right] > 0$ and so $\underline{q}_{HH} > \underline{q}_{LL}$. With $\alpha_2 = 1$ the RHS of (c3) is larger than 1 if and only if $\frac{1-\mu}{\mu}(1-\nu) > 0$, which is always the case. With $\alpha_1 = 1$ the RHS is smaller than 1, so that (c3) cannot hold. More generally, (c3) does not hold whenever $\alpha_2 = 0$. If (c3) is satisfied, then $r_{HH} > r_{LL}$. Moreover,

$$\left[S'(\underline{q}_{LL}) - S'(\underline{q}_{HH})\right] - \left[S'(\overline{q}_{HH}) - S'(\overline{q}_{LL})\right] = -2\left[1 + \frac{\nu\mu + \nu(1-\mu)(\alpha_1+\alpha_2)}{(1-\nu)(1-\mu)}\right]\Delta\theta < 0.$$

Hence, $q_{LL} > q_{HH}$ as long as S' is either linear or concave, whereas $q_{HH} > q_{LL}$ if and only if S' is sufficiently convex.

If (c3) is not satisfied, then $\underline{q}_{LL} > \underline{q}_{HH}$ so that $q_{LL} > q_{HH}$. Moreover,

$$\left[S'(\underline{q}_{HH}) - S'(\underline{q}_{LL})\right] - \left[S'(\overline{q}_{HH}) - S'(\overline{q}_{LL})\right] = -2\left[1 + \alpha_2\frac{1-\mu}{\mu} + \frac{\nu\mu + \nu(1-\mu)\alpha_2}{(1-\nu)(1-\mu)}\right]\Delta\sigma < 0.$$

With S' non-convex $r_{HH} > r_{LL}$, whereas $r_{LL} > r_{HH}$ if and only if S' is sufficiently convex.

Types HL and HH We calculate

$$\begin{aligned}
S'(\underline{q}_{HL}) - S'(\underline{q}_{HH}) &= \left[\alpha_3\frac{\nu(1-\mu)}{(1-\nu)\mu} - \frac{\nu\mu + \nu(1-\mu)(\alpha_1+\alpha_2)}{(1-\nu)(1-\mu)}\right]\Delta\theta \\
&\quad + \left[1 + \frac{1-\mu(1-\nu)}{(1-\nu)\mu} + \frac{\nu\mu + \nu(1-\mu)\alpha_2}{(1-\nu)(1-\mu)}\right]\Delta\sigma \\
S'(\overline{q}_{HL}) - S'(\overline{q}_{HH}) &= \left[\alpha_3\frac{\nu(1-\mu)}{(1-\nu)\mu} - \frac{\nu\mu + \nu(1-\mu)(\alpha_1+\alpha_2)}{(1-\nu)(1-\mu)}\right]\Delta\theta \\
&\quad - \left[1 + \frac{1-\mu(1-\nu)}{(1-\nu)\mu} + \frac{\nu\mu + \nu(1-\mu)\alpha_2}{(1-\nu)(1-\mu)}\right]\Delta\sigma
\end{aligned}$$

We have $\left[S'(\underline{q}_{HL}) - S'(\underline{q}_{HH})\right] > 0$, and hence $\underline{q}_{HH} > \underline{q}_{HL}$, if and only if

$$\nu\left[1 - 2\mu - (1-\mu)(\alpha_1+\alpha_2)\right]\frac{\Delta\theta}{\Delta\sigma} > -\left[1 - \mu + \nu\mu^2 + \nu\mu(1-\mu)\alpha_2\right]. \qquad \text{(x)}$$

It is $1 - \mu + \nu\mu^2 + \nu\mu(1-\mu)\alpha_2 > 0$, so that the RHS of (x) is negative.

(*i*) Take $2\mu + (1-\mu)(\alpha_1+\alpha_2) < 1$. Then, (x) is always satisfied so that $\underline{q}_{HH} > \underline{q}_{HL}$.

(*ii*) When $2\mu + (1-\mu)(\alpha_1+\alpha_2) > 1$, it is then convenient to rewrite (x) as

$$\frac{\Delta\theta}{\Delta\sigma} < \frac{1 - \mu + \nu\mu^2 + \nu\mu(1-\mu)\alpha_2}{\nu\left[(1-\mu)(\alpha_1+\alpha_2) - (1-2\mu)\right]}.$$

The RHS is < 1 if and only if $1 + \nu(1-\mu) < \nu\left[\alpha_1 + (1-\mu)\alpha_2\right]$. Because $1 + \nu(1-\mu) > 1$ and $\nu\left[\alpha_1 + (1-\mu)\alpha_2\right] < 1$, this is impossible. Hence, the inequality above can be satisfied. If this is the case, then $\underline{q}_{HH} > \underline{q}_{HL}$. Otherwise, the converse occurs.

We further have $[S'(\overline{q}_{HL}) - S'(\overline{q}_{HH})] > 0$, and so $\overline{q}_{HH} > \overline{q}_{HL}$, if and only if

$$\nu\left[1-2\mu-(1-\mu)(\alpha_1+\alpha_2)\right]\frac{\Delta\theta}{\Delta\sigma} > 1-\mu+\nu\mu^2+\nu\mu(1-\mu)\alpha_2. \qquad \text{(y)}$$

The RHS of (y) is positive.

(*i*) Suppose $2\mu+(1-\mu)(\alpha_1+\alpha_2)<1$. It is then convenient to rewrite (y) as

$$\frac{\Delta\theta}{\Delta\sigma} > \frac{1-\mu+\nu\mu^2+\nu\mu(1-\mu)\alpha_2}{\nu\left[1-2\mu-(1-\mu)(\alpha_1+\alpha_2)\right]}.$$

From the computations above, we know that the RHS is larger than 1, so that this condition may or may not be satisfied. We have $\overline{q}_{HH} > \overline{q}_{HL}$ if and only if it holds and the converse otherwise.

(*ii*) Take $2\mu+(1-\mu)(\alpha_1+\alpha_2)>1$. Then, (y) is never satisfied so that $\overline{q}_{HL} > \overline{q}_{HH}$.

TO BE COMPLETED