Società italiana di economia pubblica

XXII CONFERENZA

NUOVE FRONTIERE DELL'INTERVENTO PUBBLICO
IN UN MONDO DI INTERDIPENDENZA

Pavia, Università, 20-21 settembre 2010

EX ANTE VERSUS EX POST EQUALITY OF OPPORTUNITY

MARC FLEURBAEY, VITO PERAGINE

Ex Ante versus Ex Post Equality of Opportunity

Marc Fleurbaey[*], Vito Peragine[†]

Sept. 2009

Abstract

We study the difference between the ex post and the ex ante perspectives in equality of opportunity (EOp), and the possibility of a clash between them. We argue that ex ante EOp is a potential trap because someone motivated by ex post EOp may be led to believe that ex ante EOp is another natural embodiment of the same idea. As we show, it is not. Moreover, we explore the relationship between the ex post/ex ante tension and the well documented clash between the "compensation principle" and various "reward principles": we show that the tension between reward and compensation only exists if one endorses an ex post view of EOp; on the contrary, it vanishes if one adopts an ex ante view of equality of opportunity.

Keywords: equality of opportunity, ex ante/ex post, compensation, reward

JEL Classification Numbers: D63, D71.

[*]CNRS-CERSES, University of Paris-Descartes, CORE (University of Louvain-la-Neuve) and IDEP. Email: marc.fleurbaey@parisdescartes.fr.

[†]University of Bari and CHILD (University of Turin). Email: v.peragine@dse.uniba.it.

1 Introduction

The economic literature on equality of opportunity has developed concepts of fairness for the context in which individual achievements are partly the outcome of morally arbitrary circumstances (such as inherited endowments, social background...) and partly the outcome of individual effort or similar variables of personal responsibility.[1] The equality of opportunity view revolves around the idea that inequalities due to circumstances are unfair and should be eliminated as much as possible, while inequalities due to unequal effort should be considered as fair.

The economic literature on EOp that flourished in the last fifteen years has clarified that the ideal of equal opportunities is multifaceted and this appears to be the source of potential conflicts between various interpretations of the ideal and of its components. In particular, the existing literature has developed two main approaches to EOp. The *ex post* approach focuses on outcome inequalities among individuals who exert the same effort. To implement such an approach, one needs to identify the effort of individuals: this is why we call it ex post. This is the approach proposed by Roemer (1993, 1998), Fleurbaey (1995) and used by Checchi and Peragine (2009) for an empirical analysis of opportunity inequality. On the contrary, the *ex ante* approach focuses on the differences between the outcome prospects for classes of individuals with identical circumstances. Hence, the ex ante approach is more focused on *inequalities between social groups* defined by the same set of circumstances. This is the approach proposed, in different frameworks, by Van de gaer (1993) and Kranich (1996), and used in empirical studies, among others, by Bourguignon et al (2003), Ferreira and Gignoux (2008), Peragine and Serlenga (2009) and World Bank (2006).

The distinction between an ex ante situation in which circumstances are determined

[1] Book-length studies of these issues can be found in Roemer (1998) and Fleurbaey (2008).

but not yet effort and an ex post situation in which all variables are determined does not always correspond to a real time sequence, but is convenient for an intuitive interpretation of the various ethical principles. The ex ante viewpoint for compensation has to rely on some evaluation of the opportunities of individuals with unequal circumstances, and such evaluation can be inspired, as we shall see, by the metrics of opportunities that underlie the reward principles. The distinction between the ex post and the ex ante perspectives is not always clear in the existing literature: it is sometimes the case that reasearchers motivated by an ex post view of EOp may be led to believe that ex ante EOp is another natural embodiment of the same idea. As we will show in this paper, it is not. The difference between ex post and ex ante perspectives, and the possibility of a clash between them, has been hinted at in Checchi and Peragine (2005, 2009) and Fleurbaey (2008). We attempt to bring this intuition to complete fruition in this paper.

Moreover, we explore the relationship between the ex post/ex ante tension and the well documented clash (see Fleurbaey 2008) between the "compensation principle", according to which the inequalities due to circumstances should be eliminated as much as possible, and various "reward principles", which focus on how to be fair to individuals with identical circumstances and unequal effort, and how to apportion outcome to effort (once effort, by compensation, is made the sole determinant of individual success). It is possible to further distinguish between a "liberal" reward principle, which advocates submitting such equally endowed individuals to the same resource transfers, and a "utilitarian" reward principle, which suggests to have no inequality aversion for the outcomes of such individuals, and therefore to maximize the sum of their outcomes. We argue that the conflicts between the compensation principle and each of the reward principles is linked to the deeper tension between the ex post viewpoint that is implicit in the compensation idea of equalizing outcomes for individuals having exerted the same effort, and the ex ante viewpoint that is implicit in the reward idea of evaluating the treatment of individuals with equal ex ante endowments.

In this paper we propose to clarify these tensions by showing that the compensation/reward distinction and the ex post/ex ante distinction are partly independent. It is true that the reward ideas have to do with the ex ante analysis of "opportunity sets", as we will show, but one can apply compensation ideas both to the ex post and to the ex ante perspectives, with implications which are sufficiently different to generate a new clash. More precisely, we show that the tension between reward and compensation only exists if one endorses an ex post view of EOp; on the contrary, it vanishes if one adopts an ex ante view of equality of opportunity.

In this paper we concentrate on comprehensive social rankings and we leave the analysis of inequality rankings for future research. The paper is structured as follows. Section 2 introduces the formal framework. Section 3 shows the tension between ex ante and ex post perspectives on compensation. Section 4 examines the reward problem and analyzes how it relates to the ex ante/ex post tension. Section 5 concludes.

2 The Model

Individual outcomes are determined by a function $u(r, c, e)$, where r (resources), c (circumstances), and e (effort) are real numbers. The model could be generalized by assuming that r, c, e are vectors in an ordered set. This would not modify the substance of the analysis. For simplicity of the analysis that follows, effort e is only allowed to take a finite number of values (more than one) in a set $\mathcal{E}$. The function u is assumed to be continuous and strictly increasing in r and c, and not separable in (r, c), i.e., there exist r, r', c, c', e, e'

such that

$$\begin{aligned} u(r, c, e) &> u(r', c', e) \\ u(r, c, e') &< u(r', c', e'). \end{aligned}$$

An *economy* E is composed of a population, i.e., a finite or infinite set of individuals, partitioned into a finite number of "*types*" and "*cells*". A type is a set of individuals with the same circumstances c. The set of types is $T(E) = \{1, ..., n\}$, with $n \geq 2$. Let c_t denote the circumstance of type t. A cell is a set of individuals with the same characteristics (c, e). The set of cells is $C(E) = \{1, ..., m\}$, with $m > n$. The size of cell i is denoted $p(i)$ (it can be an integer for a finite population, or a real number for a continuum of individuals). With an abuse of notation (but no ambiguity), c_i can also denote the circumstance of cell i. Similarly, $p(t)$ can denote the size of type t. We use the notation $t(i)$ to identify the type containing cell i. Obviously $c_i = c_{t(i)}$.

The transfer received by cell i is denoted r_i. Formally, an economy is defined as a vector describing the profile of circumstances and effort for each cell (which induces the definition of the sets $T(E)$ and $C(E)$), as well as the size of each cell $p(.)$:

$$E = (((c_1, e_1), ..., (c_m, e_m)), p).$$

In this paper we only consider economies such that in every type t, all the set $\mathcal{E}$ is spanned by the population effort levels: for all t, $\{e \mid \exists i,\ t(i) = t \text{ and } e_i = e\} = \mathcal{E}$.

We restrict attention to anonymous transfer policies. With anonymous policies, the individuals with identical (c, e) get the same resource transfer, which defines a function $r(c, e)$ that we will call a transfer rule. Note that for every t, the function $r(c_t, .)$ is unambiguously defined over the whole set $\mathcal{E}$ thanks to the restriction made in the previous paragraph. We can define the reduced outcome function that incorporates the transfer rule:

$$f(c,e) = u(r(c,e), c, e).$$

Let u_i denote the outcome of cell $i: u_i = f(c_i, e_i)$.

Individuals belonging to type t have an opportunity set defined as the possible combinations of effort and outcome that the outcome function f makes accessible to them:

$$O_t = \{(e, f(c_t, e)) : e \in \mathcal{E}\}.$$

An opportunity distribution for the n types is denoted $\mathbf{O} = (O_1, ..., O_n)$. For a given cell i, we can also denote $O_i = O_{t(i)}$. Let $\mu(O_t)$ be the average outcome of type t :

$$\mu(O_t) = \frac{1}{p(t)} \sum_{i:t(i)=t} p(i) u_i.$$

In view of the monotonicity of u with respect to r, there is a one-to-one mapping between transfer rules r and outcome functions f, as well as between any of these and opportunity distributions. In this paper we focus on the evaluation of outcome functions f but it would be equivalent to study the evaluation of transfer rules or of opportunity distributions. A *social ordering* function defines, for every economy E in a domain $\mathcal{D}$, an ordering $\succeq(E)$ over all conceivable outcome functions, with $f \succeq(E) f'$ meaning that f is at least as good as f', and $f \succ(E) f'$ meaning that f is better than f'. The domain $\mathcal{D}$ over which these social ordering functions $\succeq(E)$ are defined is the set of economies satisfying the above conditions.

3 Compensation: ex post or ex ante

The ex post approach to compensation tries to reduce inequalities between cells having the same level of effort but different levels of outcome. The goal is to achieve a situation in which circumstances are no longer the source of inequalities. This goal is embodied in the following axiom[2], which says that it is good to reduce inequalities in outcomes between two cells sharing the same effort level but having unequal circumstances:

Ex Post Compensation: For all $E \in \mathcal{D}$, $f \succ(E)\ f'$ if there is $i, j \in C(E)$, $e_i = e_j$,

$$f'(c_i, e_i) > f(c_i, e_i) > f(c_j, e_j) > f'(c_j, e_j)$$

and $f(c_k, e_k) = f'(c_k, e_k)$ for all $k \in C(E) \setminus \{i, j\}$.

The ex ante approach to compensation seeks to identify situations of inequality based solely on information linked to the type to which individuals belong, ignoring their effort level. That is, one seeks situations in which two types are clearly unequal in terms of the perspectives offered by their circumstances and the respective transfer policies. This is the case when, as considered in the axiom below, individuals in type $t(i)$ have better circumstances than type $t(j)$, and are assured of receiving more resources. When this is observed, improving the situation of a cell i in the advantaged type while worsening that of a cell j in the disadvantaged type would worsen the situation:

Ex Ante Compensation: For all $E \in \mathcal{D}$, $f \succ(E)\ f'$ if there is $i, j \in C(E)$, $c_i > c_j$, $\min r(c_i, .) > \max r(c_j, .)$,

$$f'(c_i, e_i) > f(c_i, e_i) \text{ and } f(c_j, e_j) > f'(c_j, e_j),$$

and $f(c_k, e_k) = f'(c_k, e_k)$ for all $k \in C(E) \setminus \{i, j\}$.

This axiom is very weak, and in particular is much weaker than the next axiom, which applies when the opportunities of a given type, as depicted by the outcome function $f(c, .)$,

[2]The equity requirements in this paper are formulated in terms of Hammond's Equity Axiom. An alternative formulation in terms of the Pigou-Dalton would also be suitable and would require only simple modifications of the proofs.

dominate those of another. As domination of the outcome function can be due to the transfer rule rather than better circumstances, the following axiom covers many more situations than Ex Ante Compensation:

Srong Ex Ante Compensation: For all $E \in \mathcal{D}$, $f \succ(E)\ f'$ if there is $i, j \in C(E)$, for all $e \in \mathcal{E}$, $f(c_i, e) > f(c_j, e)$,

$$f'(c_i, e_i) > f(c_i, e_i) \text{ and } f(c_j, e_j) > f'(c_j, e_j),$$

and $f(c_k, e_k) = f'(c_k, e_k)$ for all $k \in C(E) \setminus \{i, j\}$.

The ex ante and ex post approaches to compensation, appealing though each of them may be, are incompatible.

Proposition 1 *No social ordering function defined on $\mathcal{D}$ satisfies Ex Ante Compensation and Ex Post Compensation.*

The proof is in the appendix. The incompatibility between Ex Post Compensation and Strong Ex Ante Compensation was shown in Fleurbaey (2008, ch. 9). This result is stronger and shows that even when there is no ambiguity whatsoever about the fact that a cell is better off than another on all counts in terms of their ex ante situations (better circumstances, more resources at all effort levels), reducing inequality between them may go against the goal of giving all types the same outcome function, which is encapsulated in Ex Post Compensation. Also relevant for the purpose of this paper is the fact that, as we will see in the next section, Ex Ante Compensation is sufficiently weak to be compatible with ex ante evaluations of opportunity sets that can be related to the main reward principles.

4 Reward and the evaluation of opportunities

Compensation axioms, whether they take the ex post or the ex ante standpoint, deal with the reduction of inequalities between individuals endowed with unequal circumstances. In contrast, reward principles are typically embodied in axioms that deal with individuals of the same type, in order to adjust the relationship between their outcome and their effort. We first introduce two axioms which represent the main reward principles that one finds in the literature. The liberal reward principle seeks to minimize redistribution related to differential effort levels, and therefore advocates submitting individuals with identical circumstances to equal transfers. This idea is captured by the following axiom saying that it is an improvement when the inequality in transfers received by two cells from the same type is reduced.

Liberal Reward: For all $E \in \mathcal{D}$, $f \succ(E)\ f'$ if there is $i, j \in C(E)$, $t(i) = t(j)$,

$$r'(c_i, e_i) > r(c_i, e_i) > r(c_j, e_j) > r'(c_j, e_j),$$

and $f(c_k, e_k) = f'(c_k, e_k)$ for all $k \in C(E) \setminus \{i, j\}$.

The utilitarian principle recommends an evaluation of outcome inequalities within types that is devoid of aversion to inequality, and therefore simply focuses on the sum of outcomes in order to evaluate a change affecting only one type.

Utilitarian Reward: For all $E \in \mathcal{D}$, $f \succ(E)\ f'$ if there is $i, j \in C(E)$, $t(i) = t(j)$,

$$p(i)f(c_i, e_i) + p(j)f(c_j, e_j) > p(i)f'(c_i, e_i) + p(j)f'(c_j, e_j),$$

and $f(c_k, e_k) = f'(c_k, e_k)$ for all $k \in C(E) \setminus \{i, j\}$.

The inequality in this axiom could equivalently be written: $\mu(O_i) > \mu(O'_i)$. (Note that, as $t(i) = t(j)$, one has $\mu(O_i) = \mu(O_j)$ and $\mu(O'_i) = \mu(O'_j)$). The literature has shown that each of these axioms clashes with Ex Post Compensation. In contrast, each of them is compatible with Ex Ante Compensation. We now proceed to show that there is a more basic reward axiom that underlies the two axioms. Observe that Liberal Reward expresses a strong inequality aversion with respect to transfers while Utilitarian Reward reflects zero inequality aversion with respect to outcomes. This suggests that a minimal requirement, as far as rewarding effort is concerned, is that one should evaluate changes within a type with either a positive aversion to inequality in resources or a less than infinite aversion to inequality in outcomes. In order to formulate a weak axiom reflecting this requirement, we need to define what a positive or less than infinite aversion to inequality means. Consider two real vectors (a, b) and (a', b') and let $\alpha > 0$ be a (small) real number. Let $(a, b)\ \mathsf{B}^+_\alpha\ (a', b')$ mean that $\min\{a, b\} > \min\{a', b'\}$ and

$$(a' + b') - (a + b) < \alpha\,[\min\{a, b\} - \min\{a', b'\}]\,.$$

This corresponds to a situation in which the minimum increases while the sum decreases at most a little. With a small but sufficient aversion to inequality this should be considered an improvement. Let $(a, b)\ \mathsf{B}^-_\alpha\ (a', b')$ mean that $a + b > a' + b'$ and

$$\min\{a', b'\} - \min\{a, b\} < \alpha\,[(a + b) - (a' + b')]\,.$$

This corresponds to a situation in which the sum increases while the minimum decreases at most a little. With a not too high aversion to inequality this should be viewed as an improvement.

One can then write the following axiom, which is very weak and is logically weaker than Liberal Reward and Utilitarian Reward. The axiom is restricted to cells of equal size for notational convenience (for cells with unequal sizes, one would have to define weighted variants of $\mathsf{B}^+_\alpha, \mathsf{B}^-_\alpha$).

Minimal Reward: For some $\alpha > 0$, for all $E \in \mathcal{D}$, $f \succ(E)\ f'$ if there is $i, j \in C(E)$,

$t(i) = t(j)$, $p(i) = p(j)$,

$$(r(c_i, e_i), r(c_j, e_j)) \; \mathsf{B}_\alpha^+ \; (r'(c_i, e_i), r'(c_j, e_j)),$$

$$(f(c_i, e_i), f(c_j, e_j)) \; \mathsf{B}_\alpha^- \; (f'(c_i, e_i), f'(c_j, e_j)),$$

and $f(c_k, e_k) = f'(c_k, e_k)$ for all $k \in C(E) \setminus \{i, j\}$.

Roughly, this axiom expresses the idea of priority of the worst off with respect to the allocated resources and priority of the sum with respect to the outcomes. It is extremely weak: it says that a change affecting two cells in the same type is good if the worst-off in resources is raised while the sum of resources is increased, or decreased by a sufficiently small amount, and at the same time the sum of their outcomes is increased and the worst-off in terms of outcome is raised, or decreased by a sufficiently small amount.

Nevertheless, this axiom is too much tied to the ex ante perspective, as shown in the following proposition.

Proposition 2 *No social ordering function defined on $\mathcal{D}$ satisfies Minimal Reward and Ex Post Compensation.*

The proof is in the appendix. This result is tight in the sense that if Minimal Reward were written with "For some $\alpha \geq 0$", the incompatibility would vanish. Indeed, a change would then be declared an improvement by this axiom, when $\alpha = 0$, only if the worst-off in terms of outcome were raised, making this axiom being satisfied by the leximin criterion applied to outcomes—a criterion which also satisfies Ex Post Compensation.

It is interesting to look at the connection between Ex Ante Compensation and the reward axioms, as both are based on the ex ante approach. In fact the structure of the proofs of the two propositions shows that a common underlying logic operates and can be uncovered. The utilitarian reward principle suggests that opportunities should be evaluated in terms of average outcome. One could then formulate a compensation axiom based on such evaluations, and requiring the opportunities of the least favored cell to increase—when the two cells belong to the same type, this boils down to Utilitarian Reward:

Ex Ante Utilitarian Compensation: For all $E \in \mathcal{D}$, $f \succ(E)\, f'$ if there is $i, j \in C(E)$,

$$\min\left\{\mu\left(O_{t(i)}\right), \mu\left(O_{t(j)}\right)\right\} > \min\left\{\mu\left(O'_{t(i)}\right), \mu\left(O'_{t(j)}\right)\right\},$$

and $f(c_k, e_k) = f'(c_k, e_k)$ for all $k \in C(E) \setminus \{i, j\}$.

This axiom is logically stronger than both Ex Ante Compensation and Utilitarian Reward.

It is less obvious to see what kind of metric of opportunity the liberal reward principle suggests, as Liberal Reward does not tell us how to compare individuals endowed with unequal circumstances—it only tells us that resources can be used for the comparison of individuals of the same type. One possibility is to define opportunities as would be created by the resources actually received by the cell i in the allocation under consideration:

$$O_i^r = \left\{\left(e, u\left(r_i, c_{t(i)}, e\right)\right) : e \in \mathcal{E}\right\}.$$

Indeed, if one applies this notion to cells belonging to the same type, the comparison of such opportunities is equivalent to comparing the resources their receive. And one then obtains an axiom that embodies this metric and is logically stronger than both Liberal Reward and Ex Ante Compensation:

Ex Ante Liberal Compensation: For all $E \in \mathcal{D}$, $f \succ(E)\, f'$ if there is $i, j \in C(E)$, for all $e \in \mathcal{E}$, $u\left(r_i, c_{t(i)}, e\right) > u\left(r_j, c_{t(j)}, e\right)$,

$$f'(c_i, e_i) > f(c_i, e_i) \text{ and } f(c_j, e_j) > f'(c_j, e_j),$$

and $f(c_k, e_k) = f'(c_k, e_k)$ for all $k \in C(E) \setminus \{i, j\}$.

The following table summarizes the relations between the axioms, all of which are incompatible with Ex Post Compensation:

		Ex Ante Liberal Compensation	⟶	Liberal Reward		
	↙				↘	
Ex Ante Compensation	⟵	Strong Ex Ante Compensation				Minimal Reward
		↑			↗	
		Ex Ante Utilitarian Compensation	⟶	Utilitarian Reward		

Table 1: Relation between the axioms

Table 1 shows that the tension between Compensation and Reward, in all its variants, vanishes if one adopts an ex ante view of EOp. On the other hand, all the axioms inspired by the ex ante view of EOp are incompatible with Ex Post Compensation.

5 Conclusion

The ideal of equal opportunities is multifaceted and this appears to be the source of potential conflicts between various interpretations of the ideal and of its components. In this paper we have shown that the well documented conflicts between the compensation principles and various reward principles is but an aspect of a broader conflict between ex ante and ex post perspectives. The compensation principle itself may be trapped in an internal tension between the ex ante and the ex post neutralization of inequalities in circumstances. We do not believe that such tensions and conflicts reveal an irredeemable inconsistency in the general idea of equalizing opportunity. But they do raise important ethical issues that any analyst or decision-maker interested in this approach must be aware of. If one takes the goal of providing equal opportunities as the guiding principle, and as this principle is clearly implemented only when, ex post, all individuals with the same effort obtain the equal success, we think that Ex Post Compensation must then be given priority over the other axioms. In particular, we suspect that Ex Ante Compensation (or, similarly, Strong Ex Ante Compensation) is a potential trap because someone motivated by Ex Post Compensation may be led to believe that Ex Ante Compensation (or Strong Ex Ante Compensation) is another natural embodiment of the same idea. As we have shown, it is not.

References

[1] Bourguignon F., F. Ferreira, M. Menendez 2003, "Inequality of Outcomes and Inequality of Opportunities in Brazil", Policy Research WP 3174, World Bank.

[2] Checchi D., V. Peragine 2005, "Regional disparities and inequality of opportunity: The case of Italy", IZA Discussion Paper No. 1874.

[3] Checchi D., V. Peragine 2009, "Inequality of opportunity in Italy", *Journal of Economic Inequality* (forthcoming).

[4] Kranich L. 1996, "Equitable opportunities: an axiomatic approach", *Journal of Economic Theory* 71: 131–147.

[5] Ferreira F.H.G., J. Gignoux 2008, "The measurement of inequality of opportunity: Theory and an application to Latin America", Policy Research WP 4659, World Bank.

[6] Fleurbaey M. 1995, "Three solutions for the compensation problem", *Journal of Economic Theory* 65: 505–521.

[7] Fleurbaey M. 2008, *Fairness, Responsibility, and Welfare*, Oxford: Oxford University Press.

[8] Peragine V., L. Serlenga 2008, "Equality of opportunity for higher education in Italy", in J. Bishop and B. Zheng (eds.) *Research in Economic Inequality*, Vol 12.

[9] Roemer J.E. 1993, "A pragmatic theory of responsibility for the egalitarian planner", *Philosophy and Public Affairs* 22: 146–166.

[10] Roemer J.E. 1998, *Equality of Opportunity*, Harvard: Harvard University Press.

[11] Van de Gaer D. 1993, *Equality of opportunity and investment in human capital.* Ph.D. Dissertation, Catholic University of Leuven.

[12] World Bank 2006, *World Development Report 2006: Equity and Development*, Washington, DC: The World Bank and Oxford University Press.

Appendix: Proofs

Proof of Prop. 1: By assumption there exist r, r', c, c', e, e' such that

$$\begin{aligned} u(r,c,e) &> u(r',c',e) \\ u(r,c,e') &< u(r',c',e'). \end{aligned}$$

Let an economy be composed of four types with circumstances, $c, c+\varepsilon, c', c'+\varepsilon$, where $\varepsilon > 0$ is sufficiently small so that $u(r,c,e) > u(r'+\varepsilon, c'+\varepsilon, e)$ and $u(r+\varepsilon, c+\varepsilon, e') < u(r',c',e')$.

We will focus on eight cells:

$$(c_1, e_1) = (c, e), \ (c_2, e_2) = (c, e'), \ (c_3, e_3) = (c + \varepsilon, e), \ (c_4, e_4) = (c + \varepsilon, e'),$$
$$(c_5, e_5) = (c', e), \ (c_6, e_6) = (c', e'), \ (c_7, e_7) = (c' + \varepsilon, e), \ (c_8, e_8) = (c' + \varepsilon, e').$$

Consider an allocation such that $r(c, .) \equiv r$, $r(c+\varepsilon, .) \equiv r+\varepsilon$, $r(c', .) \equiv r'$, $r(c'+\varepsilon, .) \equiv r'+\varepsilon$. Let an alternative allocation $r^*(., .)$ be derived from this one by modifying the resources given to the following cells:

$$r^*(c_1, e_1) = r + \varepsilon/3, \ r^*(c_4, e_4) = r + 2\varepsilon/3,$$
$$r^*(c_6, e_6) = r' + \varepsilon/3, \ r^*(c_7, e_7) = r + 2\varepsilon/3.$$

Compare cells 1 and 4: $c_4 = c + \varepsilon > c_1 = c$, $\min r^*(c_4, .) = r + 2\varepsilon/3 > \max r^*(c_1, .) = r + \varepsilon/3$. Therefore, by Ex Ante Compensation, the change from $r(., .)$ to $r^*(., .)$ for cells 1 and 4 (leaving all the others unaffected) is good. With the same argument, the change from $r(., .)$ to $r^*(., .)$ for cells 6 and 7 is good. By transitivity, the change from $r(., .)$ to $r^*(., .)$ is good. Compare cells 1 and 7: $e_1 = e_7 = e$ and $u_1^* > u_1 = u(r, c, e) > u(r' + \varepsilon, c' + \varepsilon, e) = u_7 > u_7^*$. Therefore, by Ex Post Compensation, the change from from $r(., .)$ to $r^*(., .)$ for cells 1 and 7 (leaving all the others unaffected) is bad. With the same argument, the change from $r(., .)$ to $r^*(., .)$ for cells 4 and 6 is bad. By transitivity, the change from $r(., .)$ to $r^*(., .)$ is bad. We have a contradiction. ■

Proof of Prop. 2: By assumption there exist r, r', c, c', e, e' such that

$$u(r, c, e) > u(r', c', e)$$
$$u(r, c, e') < u(r', c', e').$$

Let an economy be composed of two types with circumstances, c, c', and let $\varepsilon > 0$ be sufficiently small so that $u(r, c, e) > u(r' + \varepsilon, c', e)$ and $u(r + \varepsilon, c, e') < u(r', c', e')$. We will focus on four cells assumed to be of equal sizes:

$$(c_1, e_1) = (c', e), \; (c_2, e_2) = (c', e'), \; (c_3, e_3) = (c, e), \; (c_4, e_4) = (c, e').$$

Consider an allocation such that $r_1 = r' + \varepsilon$, $r_2 = r'$, $r_3 = r$, $r_4 = r + \varepsilon$. Let an alternative allocation r^* be derived from this one by modifying the resources given to these cells as follows:

$$r_1^* = r_1 - \delta, \; r_2^* = r_2 + \delta', \; r_3^* = r_3 + \gamma', \; r_4^* = r_4 - \gamma,$$

where $\delta, \delta', \gamma, \gamma'$ are chosen so that

$$\begin{aligned} u(r' + \delta', c', e') + u(r' + \varepsilon - \delta, c', e) &> u(r', c', e') + u(r' + \varepsilon, c', e) \\ \delta &< \min\{\delta'(1+\alpha), \varepsilon - \delta'\} \\ u(r + \gamma', c, e) + u(r + \varepsilon - \gamma, c, e') &> u(r, c, e) + u(r + \varepsilon, c, e') \\ \gamma &< \min\{\gamma'(1+\alpha), \varepsilon - \gamma'\}, \end{aligned}$$

where α is the parameter for which Minimal Reward is satisfied. The existence of these real numbers is easily proved. For δ and δ', pick some small $\delta' < \varepsilon$. If $\delta < \min\{\delta'(1+\alpha), \varepsilon - \delta'\}$ is small enough (but positive), it is then easy to obtain

$$u(r' + \varepsilon - \delta, c', e) > u(r' + \varepsilon, c', e) + u(r', c', e') - u(r' + \delta', c', e').$$

Let us check that we have: $(r_1^*, r_2^*) \; \mathsf{B}^+ \; (r_1, r_2)$ and $(u_1^*, u_2^*) \; \mathsf{B}^+ \; (u_1, u_2)$. The former means that $\min\{r_1^*, r_2^*\} > \min\{r_1, r_2\}$ and $r_1 + r_2 - (r_1^* + r_2^*) < \alpha\,[\min\{r_1^*, r_2^*\} - \min\{r_1, r_2\}]$. One indeed has:

$$\min\{r_1^*, r_2^*\} = \min\{r' + \delta', r' + \varepsilon - \delta\} = r' + \delta' > \min\{r_1, r_2\} = r'$$

and

$$r_1 + r_2 - (r_1^* + r_2^*) = \delta - \delta' < \alpha\delta'.$$

The latter means that $u_1^* + u_2^* > u_1 + u_2$ and $\min\{u_1, u_2\} - \min\{u_1^*, u_2^*\} < \alpha\left[u_1^* + u_2^* - (u_1 + u_2)\right]$. One indeed has

$$u_1^* + u_2^* = u(r' + \delta', c', e') + u(r' + \varepsilon - \delta, c', e) > u_1 + u_2 = u(r', c', e') + u(r' + \varepsilon, c', e)$$

and $\min\{u_1, u_2\} - \min\{u_1^*, u_2^*\} = u(r', c', e') - u(r' + \delta', c', e') < 0$. Since (r_1^*, r_2^*) B^+ (r_1, r_2) and (u_1^*, u_2^*) B^+ (u_1, u_2), it follows that Minimal Reward that changing the allocation from r to r^* for cells 1 and 2, leaving the others unchanged, is good. By a similar reasoning, one shows that changing the allocation from r to r^* for cells 3 and 4, leaving the others unchanged, is good, too. By transitivity, the change from r to r^* is good. Compare cells 1 and 3: $e_1 = e_3 = e$ and $u_1^* < u_1 = u(r' + \varepsilon, c', e) < u(r, c, e) = u_3 < u_3^*$. Therefore, by Ex Post Compensation, the change from from r to r^* for cells 1 and 3 (leaving all the others unaffected) is bad. With the same argument, the change from r to r^* for cells 2 and 4 is bad. By transitivity, the change from r to r^* is bad. We have a contradiction. ■