Società italiana di economia pubblica

XXI CONFERENZA

PUBLIC CHOICE E POLITICAL ECONOMY
I fondamenti positivi della teoria di finanza pubblica

Pavia, Università, 24-25 settembre 2009

THE GREEN SOLOW MODEL WITH LOGISTIC POPULATION CHANGE

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Abstract: We extend the Solow-Swan model with a logistic-type population growth, introduced by Ferrara and Guerrini (2008), by incorporating technological progress in pollution abatement, in a way similar to Brock and Taylor (2004). Within this framework, we determine the model's solution and prove the economy to be convergent in the long-run. In addition, we investigate sustainable growth and show that this occurs if technological progress in abatement is faster than technological progress in production. Moreover, an environmental Kuznets curve may result along the transition to the balanced growth path.

Key–Words: Green Solow, Logistic population, Environmental Kuznets curve.

1 Introduction

The traditional neoclassical model of economic growth, first developed by Solow (1956) and Swan (1956), who independently proposed similar one-sector models, provides a theoretical framework for understanding world-wide growth of output and the persistence of geographical differences in per capita output. The key concept of this model, famously known as the Solow-Swan model, is the neoclassical form of production function with declining returns to capital combined with a fixed saving rate. On the basis of these assumptions, an economy, regardless of its starting point, converges to a balanced growth path, where long-run growth of output and capital are determined solely by the rate of labor-augmenting technological progress and the rate of population growth (see, for example, Barro and Sala-i-Martin, 1995).

Ferrara and Guerrini (2008) have analyzed the role of a variable population growth rate within the Solow-Swan model by assuming a logistic-type population growth law. Within this set up, the model is proved to have a unique equilibrium, which is globally asymptotically stable. As well, its solution is shown to have a closed-form expression via Hypergeometric functions. As is typical in the neoclassical model, the human population size is assumed to be equal to the labor force. An assumption of that model, however, is that the growth rate of population is constant, yielding an exponential behavior of population size over time. Clearly, this type of time behavior is unrealistic and, more importantly, unsustainable in the very long-run. A more realistic approach would be to consider a logistic law for the population growth rate.

Brock and Taylor (2004) have demonstrated that the Solow-Swan model and the environmental Kuznets curve (hereafter EKC) are intimately related (for the EKC, see, for example, Grossman and Krueger, 1995). Amending the Solow-Swan model to incorporate technological progress in abatement, the EKC is a necessary by-product of convergence to a sustainable growth path. The resulting model, which they called the Green-Solow model, generates an EKC relationship between the flow of pollution emissions and income per capita, and the stock of environmental quality and income per capita.

The main objective of this paper is to combine within the same framework these two different research lines that have been analyzed separately in the recent past. The two research lines we aim at joining together are, respectively, the one studying the effects of including emissions, abatement and a stock of pollution in the Solow-Swan model (Brock and Taylor, 2004), and that analyzing the role of a variable population growth rate within the Solow-Swan model (Ferrara and Guerrini, 2008).

Within this framework, the economy is described by a three dimensional dynamical system, whose solution can be explicitly determined, and proved to be convergent in the long-run. Finally, we prove that sustainable growth occurs if technological progress in abatement is faster than technological progress in production. An EKC may result along the transition to the balanced growth path.

2 Model setup

We start considering the standard Solow-Swan model, i.e. a closed economy consisting of a single good that can be used either for consumption C_t or investment. Aggregate output Y_t depends on capital K_t and labor L_t according to a constant returns to scale production function. Technological progress is introduced in terms of an aggregate parameter B_t, reflecting the current state of labor-augmenting technological knowledge. Taking a Cobb-Douglas production function, we arrive at

$$Y_t = F(K_t, B_t L_t) = (B_t L_t)^{1-\alpha} K_t^{\alpha}, \qquad (1)$$

with $\alpha \in (0,1)$. The model assumes constant returns to capital K_t and effective labor input $B_t L_t$, and perfect competition. The saving rate s and the depreciation rate δ are assumed to be constant and exogenous. The evolution of capital can be described as $\dot{K}_t = sY_t - \delta K_t$, where a dot over a variable denotes differentiation with respect to time. Moreover, consumption is proportional to output, $C_t = (1-s)Y_t$.

To model the impact of pollution we follow Copeland and Taylor (1994) by assuming that pollution is jointly produced with output, and take this relationship to be proportional. Every unit of economic activity F generates Ω_t units of pollution. Pollution emitted E_t is equal to pollution created minus pollution abated. Abatement of pollution A_t takes as inputs the flow of pollution, which is proportional to the gross flow of output F, and abatement inputs, denoted by F^A. The abatement production function is standard, i.e. it is assumed to be strictly concave and having constant returns to scale. If abatement at level A_t removes the $\Omega_t A_t$ units of pollution from the total created, then we can write pollution emitted as

$$E_t = \Omega_t F - \Omega_t A(F, F^A) = \Omega_t F a(\theta), \qquad (2)$$

where $\theta = F^A/F$ is the fraction of economic activity dedicated to abatement, and $a(\theta) = 1 - A(1,\theta)$. To match the Solow-Swan model, where the exogenous technological progress in goods production raises effective labor at rate $g > 0$, i.e. $\dot{B}_t/B_t = g$, we assume exogenous technological progress in abatement lowering Ω_t at rate $g_A > 0$, i.e. $\dot{\Omega}_t/\Omega_t = -g_A$.

To combine the assumptions on pollution with our Solow-Swan model, we follow Brook and Taylor (2004) by requiring the economy to employ a fixed fraction of its inputs, both capital and effective labor, in abatement. This means that the fraction of total output allocated to abatement θ is fixed much like the familiar fixed saving rate assumption. As a result, output available for consumption or investment becomes $(1-\theta)Y_t$. In addition, we must adopt some assumption concerning natural regeneration. We treat pollution as a flow that either dissipates instantaneously, such as noise pollution, or a stock that is only eliminated over time by natural regeneration, such as sulfur or lead emissions. In other words, the stock of pollution X_t is related to the flow of emissions E_t according to

$$\dot{X}_t = E_t - \eta X_t, \qquad (3)$$

where $\eta > 0$ is the speed of natural regeneration.

The population growth rate is modelled according to Ferrara and Guerrini (2008), i.e. we have

$$\frac{\dot{L}_t}{L_t} = a - bL_t, \; a > b > 0, \qquad (4)$$

where, for simplicity, the initial population has been normalized to one, i.e. $L_0 = 1$. Usually, standard economic growth theory considers that population grows exponentially. However, this hypothesis is realistic only for an initial period, but it cannot be valid indefinitely because population growing exponentially can be arbitrarily large. What is often observed instead is that as the population grows, some members interfere with each other in competition for some critical resource. That competition diminishes the growth rate, until the population ceases to grow. It seems reasonable that a good population model must therefore reproduce this behavior. The logistic population growth model, written in equation (4), which was first investigated by Verhulst in the late 1830s, is just such a model.

Putting these assumptions together and transforming our measures of output and capital into intensive units, our modified Solow-Swan model becomes

$$\begin{aligned} \dot{k}_t &= M k_t^{\alpha} - [\delta + g + n(L_t)]\, k_t, && (5)\\ \dot{x}_t &= a(\theta)\Omega_0 e^{-g_A t} k_t^{\alpha} - [\eta + g + n(L_t)] x_t, && (6)\\ \dot{L}_t &= n(L_t) L_t, && (7) \end{aligned}$$

where $n(L_t) = a - bL_t$, $M = (1-\theta)s$, and $y_t = Y_t/B_t L_t$, $k_t = K_t/B_t L_t$, $x_t = X_t/B_t L_t$ denote output, capital and stock of pollution measured in intensive units, respectively. Given $k_0 > 0$, $x_0 > 0$, this Cauchy problem has a unique solution (k_t, x_t, L_t), defined on $[0,\infty)$ (see Birkhoff and Rota, 1978).

3 The solution

We want to provide an analytical solution for the variables appearing in the dynamical system $(5) - (7)$.

Lemma 1. *For all t, we have*

$$L_t = \frac{ae^{at}}{a - b + be^{at}}, \quad \lim_{t\to\infty} L_t = \frac{a}{b}. \qquad (8)$$

Proof: The statement is obtained by separating the L_t and t dependent parts of equation (7), integrating both sides, and then using the initial value of the population. □

Remark 2. *L_t is a monotone increasing function from $L_0 = 1$ to $L_\infty = a/b$. Moreover, $n(L_\infty) = 0$, i.e. L_∞ is a constant solution of* (7).

Proposition 3. *Let $\varphi_t = ae^{(\delta+g+a)t}/(a - b + be^{at})$. For all t, the time path of the capital stock measured in intensive units is*

$$k_t = \varphi_t^{-1}\left[k_0^{1-\alpha} + (1-\alpha)M\int_0^t \varphi_t^{1-\alpha}dt\right]^{\frac{1}{1-\alpha}} \quad (9)$$

$$\lim_{t\to\infty} k_t = \left(\frac{M}{\delta+g}\right)^{\frac{1}{1-\alpha}}. \quad (10)$$

Proof: Equation (5) is a Bernoulli differential equation. In order to solve it, we first divide the differential equation by $k_t^{-\alpha}$, and then use the substitution $z_t = k_t^{1-\alpha}$ to convert this into a differential equation in terms of z_t. The corresponding equation, $\dot{z}_t = (1-\alpha)M - (1-\alpha)\left[\delta + g + n(L_t)\right]z_t$, is a linear differential equation, whose solution we know how to determine. Since

$$e^{-(1-\alpha)\int_0^t[\delta+g+n(L_t)]dt} = e^{-(1-\alpha)(\delta+g)t}L_t^{-(1-\alpha)},$$

this solution writes

$$z_t = e^{-(1-\alpha)(\delta+g)t}L_t^{-(1-\alpha)}\cdot$$
$$\cdot\left(z_0 + (1-\alpha)M\int_0^t e^{(1-\alpha)(\delta+g)t}L_t^{1-\alpha}dt\right).$$

The first part of the statement follows by expressing the above equation in terms of k_t, and then introducing the substitution (8). In order to prove the second part of the statement, we write (9) more conveniently

$$k_t^{1-\alpha} = \frac{k_0^{1-\alpha} + (1-\alpha)M\int_0^t \varphi_t^{1-\alpha}dt}{\varphi_t^{1-\alpha}}. \quad (11)$$

φ_t is an increasing function, which diverges. Hence, $\varphi_t^{1-\alpha} \to \infty$ and $\int_0^t \varphi_t^{1-\alpha}dt \geq \int_0^t dt = t \to \infty$. Consequently, as t grows to infinity, the right-hand side of (11) leads to an indeterminate form, which can be solved applying Hopital's rule. This yields

$$\lim_{t\to\infty} k_t^{1-\alpha} = \lim_{t\to\infty}\frac{M(a-b+be^{at})}{(\delta+g)(a-b+be^{at}) + a(a-b)}.$$

The statement of (10) follows immediately. □

Remark 4. *$k_\infty = [M/(\delta+g)]^{\frac{1}{1-\alpha}}$ is a constant solution of* (5). *Moreover, $\dot{k}_t$ is positive (resp. negative) for k_t smaller (resp. larger) than k_∞. This implies that k_t decreases (increases) monotonically to k_∞ if $k_0 > k_\infty$ (resp. $k_0 < k_\infty$).*

Proposition 5. *Let $\psi_t = ae^{(\eta+g+a)t}/(a - b + be^{at})$. For all t, the time path of the stock of pollution measured in intensive units is given by*

$$x_t = \psi_t^{-1}\left(x_0 + \Omega_0 a(\theta)\int_0^t k_t^\alpha\ e^{-g_A t}\psi_t dt\right), \quad (12)$$

$$\lim_{t\to\infty} x_t = 0. \quad (13)$$

Proof: Equation (6) is a linear differential equation, which is known to be solved by

$$x_t = e^{-(\eta+g)t}L_t^{-1}\cdot$$
$$\cdot\left(x_0 + a(\theta)\Omega_0\int_0^t k_t^\alpha\ e^{(\eta+g-g_A)t}L_t dt\right).$$

The first part of the statement follows by substituting (8), and then rearranging the terms in the resulting equation. For the second part, we rewrite (12) as

$$x_t = \frac{x_0 + \Omega_0 a(\theta)\int_0^t k_t^\alpha\ e^{-g_A t}\psi_t dt}{\psi_t}. \quad (14)$$

As t grows to infinity, the denominator of the right-hand side of (14) diverges, while the numerator goes as $\int_0^\infty k_t^\alpha\ e^{-g_A t}\psi_t dt$. If this integral converges, or it is indeterminate, then (13) is immediate. Whereas, if the integral diverges, the statement follows as an application of Hopital's rule. In fact, from being

$$\lim_{t\to\infty} x_t = \lim_{t\to\infty}\frac{k_t^\alpha\ e^{-g_A t}(a-b+be^{at})}{(\eta+g+a)(a-b)+(\eta+g)be^{at}},$$

we obtain that x_t must converge to zero because k_t^α converges, and $e^{-g_A t}$ goes to zero in the long-run. □

Corollary 6. *Starting from any $k_0 > 0$, $x_0 > 0$, the long-run behavior of the model's solution is as follows: $\lim_{t\to\infty}(k_t, x_t, L_t) = \left([M/(\delta+g)]^{\frac{1}{1-\alpha}}, 0, a/b\right)$.*

4 Sustainable growth and EKC

We define a balanced growth path (BGP, hereafter) to be a trajectory along which all the relevant variables either stay constant or grow at a constant rate. We take as definition of sustainable growth a BGP generating rising consumption per capita and an improving environment.

In order to characterize the economy's balanced growth path we need the next result.

Lemma 7.

$$\gamma_{C_t} = \gamma_{Y_t} = \alpha\gamma_{K_t} + (1-\alpha)(g + \gamma_{L_t}),$$
$$\gamma_{E_t} = -g_A + \gamma_{Y_t},$$

where we have used the notation γ_z *to express the growth rate of the variable* z.

Proof: The statement follows taking logs and time derivatives of equations (1), (2), as well as recalling that consumption and output are proportional. □

Lemma 8. *Along a balanced growth path*

$$\gamma_{L_t} = 0,\ \gamma_{C_t} = \gamma_{Y_t} = \gamma_{K_t} = g,$$
$$\gamma_{X_t} = \gamma_{E_t} = g - g_A.$$

Proof: Since γ_{L_t} is constant along a BGP, we must have $L_t = a/b$. Therefore, $\gamma_{L_t} = 0$. The law of motion for capital yields $\gamma_{K_t} = sY_t/K_t - \delta$. Along a BGP, γ_{K_t} must be constant, so that Y_t/K_t must also be constant. Taking logs and time derivatives of this yields $\gamma_{Y_t} = \gamma_{K_t}$. Next, divide both sides of (3) by X_t, and note that a constant rate of change in X_t requires the ratio E_t/X_t to be constant. By log differentiation, we get that, along a BGP, the time rate of change of X_t must equal that of E_t, i.e. $\gamma_{X_t} = \gamma_{E_t}$. The statement now follows from Lemma 7. □

Remark 9. *Along a balanced growth path we must have* $\gamma_{y_t} = \gamma_{c_t} = \gamma_{k_t} = 0$, $\gamma_{x_t} = \gamma_{e_t} = -g_A$.

It follows from Lemma 8 that, along a balanced growth path, the growth rate of emissions γ_{E_t} may be positive, negative or zero. We recall that our requirements for sustainable growth imply that $\gamma_{E_t} < 0$.

Proposition 10. *There exists sustainable growth if* $g_A > g$. *Technological progress in abatement must exceed growth in aggregate output in order for pollution to fall and the environment to improve.*

Remark 11. *Brock and Taylor* (2004) *showed that* $g_A > g + n$ *is the condition for sustainable growth in case of constant population growth rate* n.

One final observation. Environmental quality deteriorates initially, and improves with economic development in later stage as the economy converges on its balanced growth path. This implies that the model produces a transition path for income per capita and environmental quality, which traces out an environmental Kuznets curve (an inverted-U shaped relationship between emissions and income). Technological progress especially in pollution abatement is primarily responsible to the inverse U-shape of this model.

5 Conclusion

In this paper, we modify an augmented Solow-Swan model by including emissions, abatement and a stock of pollution, and by assuming the population to grow according to the logistic model. In this framework, the model's solution is explicitly determined, and shown to be convergent in the long-run. For sustainable growth to be possible, technological progress in abatement has to be faster than technological progress in production. An EKC may result along the transition to the balanced growth path.

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