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BENTHAMITE FELICITY FUNCTION

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*Abstract:* This paper evaluates the effects of a Benthamite formulation for the utility function into the Ramsey model with logistic population growth, introduced by Brida and Accinelli (2007). Within this framework, we demonstrate the economy to be described by a four dimensional dynamical system, whose unique non-trivial steady state equilibrium is a saddle point with a two dimensional stable manifold. Two stable roots, rather than only one as in basic neoclassical models, now determine the speed of convergence.

*Key-Words:* Ramsey, Logistic population, Benthamite.

## 1 Introduction

The Ramsey growth model is a neoclassical model of economic growth based primarily on the work of Ramsey (1928), whose brilliant idea was to determine the saving rate endogenously, through a dynamic maximization process. This is what makes the Ramsey model different from the traditional neoclassical model of economic growth, known as the Solow-Swan model (Solow, 1956; Swan, 1956), where the saving rate is constant and exogenous. In the standard Ramsey growth model, the human population size is assumed to be equal to the labor force. An assumption of that model, however, is that the growth rate of population is constant, yielding an exponential behavior of population size over time. Clearly, this type of time behavior is unrealistic and, more importantly, unsustainable in the very long-run. A more realistic approach would be to consider a logistic law for the population growth rate. This approach was considered by Brida and Accinelli (2007), who analyzed how the Ramsey model is affected by the choice of a logistic growth of population. Their analysis was done under the assumption that the society's welfare is measured by a utility function of per capita consumption. However, an alternative plausible formulation of the planner's problem would be to use the so-called Benthamite welfare function, in which society's welfare over time is measured by weighting the utility index of per capita consumption by numbers, i.e. multiplying the utility function of the representative man by the total population. This is exactly the main objec-

tive of our paper. In our analysis of the Ramsey model with logistic population growth under the Benthamite formulation, we see that the economy is described by a dynamical system, whose unique non-trivial steady state equilibrium is saddle point stable. As well, the stable saddle-path is two dimensional, thus enriching the transitional adjustment paths relative to that of the standard Ramsey growth model. Two stable roots, rather than only one as in basic neoclassical models, now determine the speed of convergence.

## 2 Model setup

Consider a closed economy populated by a fixed number of identical infinitely lived households that, for simplicity, is normalized to one. Following Brida and Accinelli (2007), the household size,  $L_t$ , grows according to the logistic growth law

$$\dot{L}_t = L_t(a - bL_t), \quad a > b > 0, \quad (1)$$

where  $L_0$  is normalized to one, a dot over a variable denoting time derivative. Let  $C_t$  be aggregate consumption, and  $c_t = C_t/L_t$  denote consumption per capita. Each household maximizes its dynastic utility

$$\int_0^{\infty} \frac{c_t^{1-\theta}}{1-\theta} L_t e^{-\rho t} dt, \quad (2)$$

where  $\rho > 0$  is the rate of time preference, and  $\theta > 0$  represents the inverse of the elasticity of intertemporal

substitution. Contrary to Brida and Accinelli (2007), the felicity function  $c_t^{1-\theta}/(1-\theta)$ , known as constant intertemporal elasticity of substitution (or CIES) function, is multiplied by the size of the family, indicating that at any point in time overall utility is equal to the addition of the felicities of all family members alive at that time. This means that the felicity function becomes  $L_t c_t^{1-\theta}/(1-\theta)$ , the so-called Benthamite welfare function, so that the number of family members receiving the given utility level is taken into account. Output  $Y_t$  is produced with the Cobb-Douglas technology

$$Y_t = K_t^\alpha L_t^{1-\alpha}, \quad \alpha \in (0, 1),$$

where  $K_t$  denotes the capital stock. The household's budget constraint is  $Y_t = I_t + C_t$ , where  $I_t$  is gross investment. The capital stock accumulates according to the following law of motion

$$\dot{K}_t = I_t - \delta K_t,$$

where  $\delta > 0$  is the depreciation rate. Let  $y_t = Y_t/L_t$  and  $k_t = K_t/L_t$  denote output and capital stock per capita, respectively. The production function can be expressed in intensive form as  $y_t = k_t^\alpha$ . As well, taking derivatives with respect to time in the definition of  $k_t$ , the budget constraint becomes

$$\dot{k}_t = k_t^\alpha - (\delta + a - bL_t)k_t - c_t. \quad (3)$$

The household's optimization problem is to maximize its dynastic utility (2) subject to constraints (1) and (3). Solving this continuous-time dynamic problem involves using calculus of variations. Let  $H$  be the current-value Hamiltonian of the household's problem, i.e.

$$H = \frac{c_t^{1-\theta}}{1-\theta} L_t + \lambda_t [k_t^\alpha - (\delta + a - bL_t)k_t - c_t] + \mu_t [L_t(a - bL_t)],$$

where  $\mu_t$  and  $\lambda_t$  are the costate variables associated to (1) and (3), respectively. The Pontryagin conditions for optimality are given by  $H_{c_t} = 0$ ,  $\dot{\lambda}_t = \rho\lambda_t - H_{k_t}$ ,  $\dot{\mu}_t = \rho\mu_t - H_{L_t}$ ,  $\dot{k}_t = H_{\lambda_t}$ ,  $\dot{L}_t = H_{\mu_t}$ , together with the transversality conditions. These yields

$$c_t^{-\theta} L_t = \lambda_t, \quad (4)$$

$$\dot{\lambda}_t = -\lambda_t [\alpha k_t^{\alpha-1} - \delta - \rho - (a - bL_t)],$$

$$\dot{\mu}_t = \mu_t (\rho - a + 2bL_t) - b\lambda_t k_t - \frac{c_t^{1-\theta}}{1-\theta},$$

plus equations (1) and (3), as well as

$$\lim_{t \rightarrow \infty} e^{-\rho t} \lambda_t k_t = 0, \quad \lim_{t \rightarrow \infty} e^{-\rho t} \mu_t L_t = 0.$$

Differentiating (4) with respect to time, and using formula (4), we can rid (4) of the  $\dot{\lambda}$  and  $\lambda$  expressions. After rearrangement, we get that the dynamic behavior of the economy can be described by the following system of differential equations

$$\dot{k}_t = k_t^\alpha - (a - bL_t + \delta) k_t - c_t,$$

$$\dot{c}_t = \frac{c_t}{\theta} (\alpha k_t^{\alpha-1} - \delta - \rho),$$

$$\dot{L}_t = L_t(a - bL_t),$$

$$\dot{\mu}_t = \mu_t (\rho - a + 2bL_t) - c_t^{-\theta} \left( bL_t k_t + \frac{c_t}{1-\theta} \right),$$

together with the following conditions

$$\lim_{t \rightarrow \infty} e^{-\rho t} c_t^{-\theta} L_t k_t = 0, \quad \lim_{t \rightarrow \infty} e^{-\rho t} \mu_t L_t = 0.$$

**Remark 1.** *Compared with the model of Brida and Accinelli (2007), population growth has now no effect on the growth rate of per capita consumption.*

### 3 Local dynamics

We now proceed to carry out the study of local dynamics of the above dynamical system. We focus on the steady state at which the growth rates of  $k_t$ ,  $c_t$ ,  $L_t$  and  $\mu_t$  are equal to zero. Our analysis is restricted to the case of interior steady states in order to exclude the economically meaningless solutions such as  $k_* = 0$ ,  $c_* = 0$ , or  $L_* = 0$ . An asterisk below a variable denotes its stationary value. We can state the following result.

**Lemma 2.** *The unique non-trivial steady state of the economy is*

$$k_* = \left( \frac{\alpha}{\delta + \rho} \right)^{\frac{1}{1-\alpha}}, \quad c_* = \left[ \frac{\rho + (1-\alpha)\delta}{\alpha} \right] k_*,$$

$$L_* = \frac{a}{b}, \quad \mu_* = \frac{c_*^{-\theta}}{a + \rho} \left( a k_* + \frac{c_*}{1-\theta} \right).$$

*Proof.* Imposing the stationary conditions  $\dot{k}_t = \dot{c}_t = \dot{L}_t = \dot{\mu}_t = 0$  yields the equations

$$k_t^\alpha - \delta k_t = c_t, \quad \alpha k_t^{\alpha-1} = \delta + \rho, \quad L_t = \frac{a}{b},$$

$$\mu_t (a + \rho) - a c_t^{-\theta} k_t - \frac{c_t^{1-\theta}}{1-\theta} = 0.$$

The steady state value can now be determined in a recursive manner.  $\square$

**Proposition 3.** *The steady state is a saddle point with a two dimensional stable manifold.*

*Proof.* From the theory of linear approximation, we know that in a neighborhood of the steady state the dynamic behavior of a non-linear system is characterized by the behavior of the linearized system around the steady state. In our case this means

$$\begin{bmatrix} \dot{k}_t \\ \dot{c}_t \\ \dot{L}_t \\ \dot{\mu}_t \end{bmatrix} = J^* \begin{bmatrix} k_t - k_* \\ c_t - c_* \\ L_t - L_* \\ \mu_t - \mu_* \end{bmatrix}, \quad (5)$$

where  $J^* = (J_{ij}^*)$ ,  $i, j = 1, 2, 3, 4$ , is the Jacobian matrix evaluated at the steady state  $(k_*, c_*, L_*, \mu_*)$ . By definition,  $J_{11}^* = (\partial \dot{k}_t / \partial k_t)_*$ ,  $J_{12}^* = (\partial \dot{k}_t / \partial c_t)_*$ ,  $J_{13}^* = (\partial \dot{k}_t / \partial L_t)_*$ ,  $J_{14}^* = (\partial \dot{k}_t / \partial \mu_t)_*$ . Similarly for the other  $J_{ij}^*$  entries. Computing these elements, we get

$$J_{11}^* = \rho, \quad J_{12}^* = -1, \quad J_{13}^* = bk_*, \quad J_{14}^* = 0,$$

$$J_{21}^* = -\frac{(1-\alpha)\alpha c_* k_*^{\alpha-2}}{\theta}, \quad J_{22}^* = J_{23}^* = J_{24}^* = 0,$$

$$J_{31}^* = J_{32}^* = J_{34}^* = 0, \quad J_{33}^* = -a,$$

$$J_{41}^* = -ac_*^{-\theta}, \quad J_{42}^* = \theta c_*^{-1} \mu_* (a + \rho) - \frac{c_*^{-\theta}}{1-\theta},$$

$$J_{43}^* = 2b\mu_* - bk_* c_*^{-\theta}, \quad J_{44}^* = a + \rho,$$

Two eigenvalues of  $J^*$  are immediately seen to be given by  $\xi_1 = a + \rho$ ,  $\xi_2 = -a$ , and the other two eigenvalues are those of the submatrix  $\begin{pmatrix} J_{11}^* & J_{12}^* \\ J_{21}^* & J_{22}^* \end{pmatrix}$ . The two roots of the characteristic equation associated with this matrix are

$$\xi_{3,4} = \frac{\rho}{2} \pm \sqrt{\frac{\rho^2}{4} + A},$$

where we define  $A = (1-\alpha)\alpha c_* k_*^{\alpha-2} / \theta$ . Recalling that the determinant (resp. trace) of a matrix is also equal to the product (resp. sum) of its eigenvalues, we derive that these roots are real with opposite signs. Let  $\xi_3$  be the smaller and  $\xi_4$  be the bigger root. In conclusion, we have found that the matrix  $J^*$  has two real positive (unstable) and two real negative (stable) roots. This proves that the steady state is (locally) a saddle point. The stable manifold is the hyperplane generated by the associated eigenvectors, with dimension equal to the number of negative eigenvalues (see Simon and Blume, 1994).  $\square$

From (5) we see that the system

$$\begin{bmatrix} \dot{k}_t \\ \dot{c}_t \\ \dot{L}_t \end{bmatrix} = \begin{bmatrix} \rho & -1 & bk_* \\ -A & 0 & 0 \\ 0 & 0 & -a \end{bmatrix} \begin{bmatrix} k_t - k_* \\ c_t - c_* \\ L_t - L_* \end{bmatrix}$$

describes the behavior of  $k_t, c_t, L_t$  around the steady state values. The solution to such a linear system is known to be given by

$$\begin{cases} k_t - k_* = B_1 d_{11} e^{\xi_2 t} + B_2 d_{12} e^{\xi_3 t} + B_3 d_{13} e^{\xi_4 t}, \\ c_t - c_* = B_1 d_{21} e^{\xi_2 t} + B_2 d_{22} e^{\xi_3 t} + B_3 d_{23} e^{\xi_4 t}, \\ L_t - L_* = B_1 d_{31} e^{\xi_2 t} + B_2 d_{32} e^{\xi_3 t} + B_3 d_{33} e^{\xi_4 t}, \end{cases}$$

where  $B_1, B_2, B_3$  are arbitrary constants, to be determined using the initial conditions and the transversality conditions, and the vectors  $[d_{11} \ d_{21} \ d_{31}]^T$ ,  $[d_{12} \ d_{22} \ d_{32}]^T$ ,  $[d_{13} \ d_{23} \ d_{33}]^T$  are the eigenvectors associated with each of the three roots  $\xi_2, \xi_3, \xi_4$ . The eigenvectors associated with  $\xi_i$  ( $i = 2, 3, 4$ ) are obtained solving

$$\begin{bmatrix} \rho - \xi_i & -1 & bk_* \\ -A & -\xi_i & 0 \\ 0 & 0 & -a - \xi_i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0.$$

So that we get

$$\begin{cases} k_t - k_* = B_1 abk_* e^{-at} + B_2 \xi_3 e^{\xi_3 t} + B_3 \xi_4 e^{\xi_4 t}, \\ c_t - c_* = B_1 Abk_* e^{-at} - B_2 Ae^{\xi_3 t} - B_3 Ae^{\xi_4 t}, \\ L_t - L_* = B_1 [A - (\rho + a)a] e^{-at}. \end{cases}$$

Because  $e^{\xi_4 t}$  diverges to infinity, convergence to the steady state immediately implies  $B_3 = 0$ . Thus, the solutions along the stable manifold of the saddle-path are given by

$$\begin{cases} k_t - k_* = B_1 abk_* e^{-at} + B_2 \xi_3 e^{\xi_3 t}, \\ c_t - c_* = B_1 Abk_* e^{-at} - B_2 Ae^{\xi_3 t}, \\ L_t - L_* = B_1 [A - (\rho + a)a] e^{-at}, \end{cases} \quad (6)$$

with the constant appearing in the solution obtained from the initial conditions.

## 4 Speed of convergence

In previous growth models, in which all variables moved in proportion to one another, the associated unique stable eigenvalue sufficed to characterize the transition. With two stable roots,  $\xi_2$  and  $\xi_3$ , the speeds of adjustment change over time, although asymptotically all scale-adjusted variables converge to their respective equilibrium at the rate of the slower growing eigenvalue,  $\min\{-\xi_2, -\xi_3\}$ . In general, we define the speed of convergence at time  $t$  of a variable  $z$

as

$$\beta_t = \frac{\dot{z}_t}{z_t - z_*}. \quad (7)$$

This expression measures the rate of convergence at any instant of time in terms of the percentage rate of change in the distance  $z_t - z_*$ . When the stable manifold is one dimensional, this measure equals the magnitude of the unique stable eigenvalue. By contrast, in the present model, the stable transitional path is a two dimensional locus, and a two-dimensional stable manifold generates time-varying convergence speeds. It is nevertheless desirable to have one comprehensive measure, that summarizes the speed of convergence of the overall economy. For this purpose the percentage change in the Euclidean distance

$$\beta_t = \sqrt{(k_t - k_*)^2 + (c_t - c_*)^2 + (L_t - L_*)^2} \quad (8)$$

serves as a natural summary measure of the speed of convergence. At any instant of time, the generalized speed of convergence is a weighted average of the speeds of convergence of three variables, the weights being the relative square of their distance from the steady state equilibrium. As well, note that log differentiation of (8) yields

$$\frac{\dot{\beta}_t}{\beta_t} = \sum_{z \in \{k, c, L\}} \left[ \frac{(z_t - z_*)^2}{\beta_t^2} \right] \frac{\dot{z}_t}{z_t - z_*}. \quad (9)$$

Therefore, we derive that (8) generalizes the one dimensional measure (7). Finally, (6) implies that the speed of convergence of any variable at any point in time is a weighted average of the two negative eigenvalues of  $J^*$ . Over time, the weight of the smaller (more negative) eigenvalue declines, so that the larger of the two stable (negative) eigenvalues describes the asymptotic speed of convergence.

## 5 Conclusion

In this paper we have considered a modified version of the standard Ramsey growth model, obtained by introducing a Benthamite utility function, and a logistic-type population growth law. This set up has led the model to be described by a four dimensional dynamical system, that is proved to have a unique non-trivial steady state equilibrium (a saddle point). The saddle-path stable system now has two negative eigenvalues, so that the stable manifold is a two dimensional locus, thereby introducing important flexibility to the convergence and transition characteristics. The crucial determinant of the asymptotic speed of convergence is the larger of the two negative eigenvalues.

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