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HIGHLIGHTING THE SOLUTION FOR THE ECONOMIC LIFE OF FIXED CAPITAL IN SRAFFA'S SYSTEM UNVEILS NEW MICROECONOMIC FOUNDATIONS FOR ECONOMIC THEORY

GIUSEPPE VITALETTI
Tuscia University, Viterbo, Italy

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Highlighting the Solution for the Economic Life of Fixed Capital in Sraffa's System Unveils New Microeconomic Foundations for Economic Theory

Giuseppe Vitaletti
Tuscia University, Viterbo, Italy

Determining the optimal life of capital has received little attention in economic literature. It is clear, however, that in practice, this is an important issue, especially when the necessity to consider the life of capital as variable is obvious, as in the case of shifts adoption. This paper deals with such a problem, starting the analysis from Sraffa's model of fixed capital. By changing some fundamental aspects of its frame, a solution is worked out, which shows very interesting properties. In particular, for the price determination of commodities, the importance of an equation emerges, where only intermediate input and labor costs appear. Some parts of the marginalist theory of production are then integrated into the model, while other parts are strongly criticized. In the end some possible applications of the new scheme are indicated, also at the macroeconomic level.

1. Introduction

A few years after the publication of Piero Sraffa's *Production of Commodities by Means of Commodities* (1960), the incongruity in taking the number of years of useful life of machines as given was pointed out, relatively to the chapter on fixed capital. Some Sraffian economists have worked on the problem and have elaborated possible solutions.

This paper presents the results of a specific approach, worked out when, in studying the economic effects of fiscal measures to change the yearly number of plant operating hours in the Sraffian context, I came across the above-mentioned difficulty.

In paragraph 2 a first review of the problem, using Sraffa's system of equations, is offered, and the past solutions are briefly commented on.

In paragraph 3 the analysis is developed, also in formal terms, starting from the simplified case of the production of a non-basic commodity (an interpretation of which is delivered in terms of traditional partial equilibrium models for price determination), and assuming the rate of profit at a given level. Constant efficiency and decreasing efficiency are then considered, the latter in the simplified hypothesis of increasing variable costs as the plant gets older. Many formal methods to calculate the solution for the optimal life of plants are presented, together with illustrative examples, and the difference with past approaches is made clear, in

I wish to thank the discussants of preliminary versions of this work at: a) the XIII° International Conference on Input-Output Techniques, held at the University of Macerata on 21-25 August, 2000; b) the meeting at the Centro Sraffa, University III of Rome, carried out on 14 June, 2002. Particular thanks also to Sergio Steve, for having stimulated, thirty years ago, the research on fiscal measures to change the operating hours of plants, the premise of this work.

particular with the one called “truncation”.

In paragraph 4 the limitations of the foregoing analysis are removed. Firstly (in 4.1), this is done within Sraffa’s framework, precisely by:

a) introducing the possibility of increasing efficiency;
b) enlarging the field of conditions that may result in decreasing (and increasing) efficiency, in particular by allowing that output varies with the age of the plant;
c) analyzing the case of basic products;
d) considering the possibility that the rate of profit assumes different levels.

Further extensions, that go beyond Sraffa’s framework, are treated in 4.2. They regard:

e) technical progress;
f) the possibility that a plurality of capital goods is used for the production of a single commodity;
g) the variability of the relation between costs and the quantity produced in each operating period.

In the final section of paragraph 4, the possibility to integrate some aspects of the marginalist theory into a deeply modified Sraffian framework emerges. A few potential developments of this radically new approach are indicated in the last paragraph.

2. A first insight into the problem

2.1 *Sraffa’s framework revisited*

Before starting the analysis, first let us review how Sraffa treats fixed capital within his system, with the following set of equations[1]:

System 1
Sraffa’s framework for the treatment of fixed capital

$$
\begin{aligned}
(M_1 p_{m1} + A_{g1} p_a + \cdots + K_{g1} p_k)(1+r) + L_{g1} w &= G_1 p_g + M_2 p_{m2} \\
(M_2 p_{m2} + A_{g2} p_a + \cdots + K_{g2} p_k)(1+r) + L_{g2} w &= G_2 p_g + M_3 p_{m3} \\
&\cdots\cdots \\
(M_i p_{mi} + A_{gi} p_a + \cdots + K_{gi} p_k)(1+r) + L_{gi} w &= G_i p_g + M_{i+1} p_{m,i+1} \\
&\cdots\cdots \\
(M_{n-1} p_{m,n-1} + A_{g,n-1} p_a + \cdots + K_{g,n-1} p_k)(1+r) + L_{g,n-1} w &= G_{n-1} p_g + M_n p_{mn} \\
(M_n p_{mn} + A_{gn} p_a + \cdots + K_{gn} p_k)(1+r) + L_{gn} w &= G_n p_g
\end{aligned}
$$

[1] *System* 1 is taken from Sraffa (1960, p. 65). The only differences are: a) the advancement by one year of the subscripts designating machine age, thus stressing the period of operation rather than its age at the beginning of the year; b) the explicit provision for the possibility that the quantity of input, of labor, and of output, may vary from year to year, which Sraffa considers but does not formalize.

The equations describe the separate production processes 1, 2, ... *n,* that correspond to the operating periods of the machine. The target of the processes is the attainment of output *G* (the quantity of which may vary depending on the year of production), carrying price p_g. The current operating costs consist of the intermediate inputs A_g ... K_g, with prices p_a ... p_k, and of labor, L_g, to which corresponds the unit wage *w*. Again, the quantities of inputs and labor may vary with the year of production.

Fixed capital appears in the equations both as input, indicated as quantity M_1 with price p_{m1} when the machinery is new and in its first year of operation, as quantity M_2 with price p_{m2} when it is one year old and in its second year of operation, and as quantity M_n with price p_{mn} when it is n - 1 years old and in its last year of operation. It appears as output as well, indicated by the same quantities and prices as appear as input in the following year - except the last, when, as the capital does not go into production the following year, it is not found on the output side. The advanced costs, consisting of the fixed capital and intermediate inputs, are passed onto output price through application of the profit rate *r*.

Let us start from the special case that *G* is a non-basic product. As will be shown, this assumption does not affect any of the main conclusions, but it makes the price of machines in their first year of operation (p_{m1}), and also the prices of the intermediate inputs (p_a, p_b,... p_k), independent from p_g, which greatly simplifies the analysis. If we posit *w* as the *numeraire,* and *r* as a given and exogenous parameter (as Sraffa does), it is clear that *System* 1 can determine the *n* unknowns $p_{m2}, p_{m3}, \dots p_{m,i+1}, \dots p_{mn}, p_g$.

Firstly we consider the case of r=0 in a context of constant efficiency, which in *System* 1 implies that the physical quantity of intermediate input, of labor and of output, are the same for all the equations (i.e. for all the years of production). It is clear that, if the price p_g must be the same for each equation and the value of the capital has to satisfy the condition of reaching zero at the end of the process, the only possibility is that capital assumes a quantitative determination on the left which differs from that on the right by the same amount in all the equations, and that this difference is equal to M_1p_{m1} divided by *n*. When r>0, it still holds that the contribution of capital on the cost side (equal to its value multiplied by 1+*r*) must exceed that on the revenue side by the same amount in each equation, but in this circumstance the first term comprises an interest component[2]. Consequently in the upper part of *System* 1 the difference between the value of capital in the left and right-hand sides of the equations will be smaller than M_1p_{m1} divided by *n*, whereas this inequality is reversed in the equations in the lower part of the system. Correspondingly, the interest component decreases, since the rate *r* is applied to progressively diminishing values of capital. *But in whatever case,* (r= 0 or r>0), *the logical process adopted precludes the value of fixed capital (and therefore its price)*

[2] Obviously when r>0 the set of prices p_{m1}, p_a,... p_k taken by the general system is different from the case r=0.

from becoming negative after a certain time-span. In other terms, whatever equation is considered, the price of the capital is positive.

Things change if we abandon the hypothesis of constant efficiency for the more realistic one of decreasing efficiency, which implies that current operating costs (intermediate inputs and labor) increase, and/or that the quantity of output shows reductions, when in *System* 1 we pass to the equations denoting further years of production. Again we can start from the case of r=0, with decreasing efficiency expressed simply in the form of yearly increments of current operating costs. It is clear that the decrease in the value of fixed capital from M_1p_{m1} towards zero must accommodate such increments, in the sense that, in each equation, the difference between the capital value on costs and revenues sides must “leave room” for them. Specifally, the sum of:

a) the rise in current operating costs;
b) the difference of left-hand and right-hand capital values;

must be the same for each equation, otherwise the price of output G differs with the year of production. At the same time we have to maintain the constraint that the sum which refers to the differences of capital values through all the equations of *System* 1 is equal to M_1p_{m1}, in order to assure the total depletion of the value of the capital in the time-span of the productive life of the plant.

Suppose we choose a certain n (let us say n_0), which corresponds to n_0 equations and, as noted, to only one set of possible prices for the various aged equipment. This implies in particular a determinate price p_{m2} for M_2, which brings about a specific value for the difference with respect to M_1p_{m1} in the first equation. If this difference proves to exceed the sum of the cumulative increments in current operating costs calculated between the year n_0 and the first year, it follows that in the last year there is still room for a positive quantitative determination of depletion. This means that the value of the equipment in the end reaches zero starting from a positive value on the left-hand side of the last equation. Everything seems to work but there is a problem. Had we posited $n_0' > n_0$ (with n_0' not much greater than n_0), we would probably arrive at a solution with positive depreciations[3], and still with a positive value of the equipment on the cost side in the last equation, but with a smaller value for the depletion in the first year. This would be due to the greater number of equations, which allows the firm to spread the same total depletion over a longer period. *In this case the choice of* n_0' *would be more convenient*, since the price of output G would be less than with n_0 (as is clear from the first equation, where a greater value for M_2p_{m2} results in presence of the same current operating costs as in the choice of n_0).

If we choose for the duration of the plant a value of n significantly greater than n_0 (let us call it n_1), the difference between M_1p_{m1} and M_2p_{m2} should be lower than in n_0, if the depletion in the years between n_1 and n_0 bears positive quantitative determinations. In the same choice, however, the sum of the cumulative increments in

[3] Henceforth, the terms “depreciation” and “depletion” will be interchangeable. Later the same concept will also be referred to by the term “amortization”.

current operating costs calculated between the year n_1 and the first year is normally much greater than that calculated in correspondence with n_0, since with decreasing efficiency such costs have been assumed as increasing over time. On these positions we should expect that for some determinations of n_1 the latter magnitude exceeds the difference between M_1p_{m1} and M_2p_{m2}. But this outcome is mathematically and economically impossible, since it would imply that the price of the product G in the last equation is greater than in the first one. The only possibility in this case is that in some equations, in particular the ones corresponding to the values of n near to n_1, depreciation is negative. Moreover, given that the algebraic sum of depreciations must be M_1p_{m1}, the terms referring to the years with a positive sign total up more than M_1p_{m1}, and so the value of capital on the right must become zero (and then negative) in years which precede the ones in which the annual depletion becomes negative[4]. As will be apparent in paragraph 3, although already intuitively obvious at this stage of the analysis, the picture does not change significantly with: (a) the introduction of more complex hypotheses on efficiency; (b) the removal of the position r=0; (c) the consideration of non-basic production.

2.2 *The attempted solutions*

Regarding the economic literature, the first attempt to tackle the above mentioned problem is "truncation", which consists of a process of progressive elimination in *System* 1 of the equations for which negative values of depreciation and/or of capital appear, starting from the last one. Clearly, in this way, a value $n_1' < n_1$ will be reached which shows only positive prices. Nevertheless, it can be observed that truncation is an empirical and incomplete technique. According to Schefold (1980, p. 179), perhaps the main author on this theme, negative capital prices indicate the necessity of truncation, which entails the elimination of a certain number of equations. But positive prices may not represent a satisfactory solution. The opposite of truncation, i.e. the addition of equations to the system, may, indeed, bring about a better solution, in terms of lower prices (as we have pointed out above, in paragraph 2.1).

Baldone (1980, pp. 102-106) and Kurtz & Salvadori (1995, Chapter 7) elaborate in this direction, by treating the problem of plant life in the context of entrepreneurial decisions among alternative choices. Their view entails a criticism of

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[4] An illustrative example will be provided in paragraph 3.5 (*Table* 2). It should be intuitively clear from the previous analysis that:

a) if the duration of capital is fixed independently from prices, as done by Sraffa, in any case, given the constraint of total amortization of the initial investment, the value of the capital on the right-hand side of the system in the last equation must be zero;
b) only the determinations of the duration in which the capital value is non-negative for each equation are to be considered;
c) if the duration has to be chosen as to minimize the annual total cost and therefore the price of the product, it must show the characteristic that, in the equation corresponding to it, amortization tends towards zero. In fact, in the case of positive prices, it is convenient to increase the duration up to the point at which the room for a positive amount of amortization in the last period is as subtle as possible, tending as limit towards zero. This is true since, within the respect of the constraint of positivity, the greater the number of equations the lower the amortization in the first equation, and as a consequence the lower is the annual total cost.

In particular statement c), which implies that the value of capital must tend to zero in the last equation in the left-hand side of the system as well as in the right-hand side, condensates the core of this work.

Sraffa's framework, which is more significant than truncation, since, unlike in *Production of Commodities by Means of Commodities,* fixed capital is considered in the context of the analysis of optimal pricing - a characteristic that, on the contrary, is not very clear in the truncation approach.

All these developments of Sraffa's original treatment of fixed capital are formally correct but overlook some fundamental aspects, particularly consideration of specificities in economic terms of the equation to which optimal life of the plant corresponds (c.f. footnote 4 for a first insight, and paragraph 3.4, 3.5 and 4.2.1 for further specific observations). Highlighting these characteristics, also in formal terms, represents the core of the next paragraph.

3. The characteristics of the solution for the optimal life in simplified hypotheses

3.1 *Non-basic production and traditional concepts in microeconomic analysis*

We will maintain in this paragraph all the simplifying hypotheses adopted in par. 1.1. Moreover, in what follows the circumstance that G is a non-basic product will be "exploited" to change the notation and the presentation of the magnitudes of *System* 1, making them comparable with the usual concepts of traditional microeconomic analysis: mainly investment, related fixed annual costs, and variable costs.

Specifically, let us start by indicating the cost of the initial purchase of the plant ($M_1 p_{m1}$) with the unique symbol I; the value amount of intermediate goods with the unique symbol Vg (variable costs related to intermediate goods); and the labor cost with Vl (variable costs related to labor). This notation is in fact allowed by the hypotheses of non-basic production and of r given, which make the values I, Vg and Vl independent from p_g.

Furthermore, we introduce in *System* 1 some formal changes aimed at improving its capacity to represent the way fixed capital is viewed in the real economy by accountants and entrepreneurs. In particular, to accomplish this target, we isolate as output, on the right-hand side of equations, the product G alone, switching the capital values to the left-hand side of the system. Thus in the first equation the left-hand side, in the part regarding capital, becomes $M_1 p_{m1} (1 + r) - M_2 p_{m2}$: an expression which can be transformed into $Ir + D_1$, taking into account what has been posited above and denoting with D_1 the difference in the machinery's value between the start of the first and of the second year. Similarly, in the second equation, instead of $M_2 p_{m2} (1 + r) - M_3 p_{m3}$, we can posit $(I - D_1) r + D_2$. In general, *System* 1 can be transformed into the following *System* 2:

System 2
System 1 *revisited for the case of non-basics*

$$
\begin{array}{llllllllll}
I\,r & + & D_1 & + & Vg_1\,(1+r) & + & Vl_1 & = & G_1\,p_g \\
(I - D_1)\,r & + & D_2 & + & Vg_2\,(1+r) & + & Vl_2 & = & G_2\,p_g \\
\cdots\cdots & & & & & & & & \\
(I - D_1 - \cdots - D_{i-1})\,r & + & D_i & + & Vg_i\,(1+r) & + & Vl_i & = & G_i\,p_g \\
\cdots\cdots & & & & & & & & \\
(I - D_1 - \cdots - D_{n-2})\,r & + & D_{n-1} & + & Vg_{n-1}(1+r) & + & Vl_{n-1} & = & G_{n-1}\,p_g \\
(I - D_1 - \cdots - D_{n-1})\,r \ldots & + & D_n & + & Vg_n\,(1+r) & + & Vl_n & = & G_n\,p_g
\end{array}
$$

where, in the last equation, D_n *is equivalent to* $I - D_1 - ..- D_i - ..- D_{n-1}$, *given the constraint that the sum of annual depreciation must be equal to the initial capital value, I*: so the *n* unknowns of *System* 2 are D_1 , D_2 ,...D_{n-1}, p_g . The sum of the first two terms in each equation represents the related fixed annual cost to the initial investment (that will be denoted by F_i for the generic year *i*), broken down into its interest component (the first addend) and depreciation or amortization (second addend)[5].

Note that, at this point, the framework we are working within (values of machines, intermediate input and labor taken as given; price of output established without considering the interaction with capital and intermediate input values; rate of profit set at a certain level, corresponding to that prevailing in the economic system; capital which enters only the cost side of production, through the two "fixed" components, interest and amortization), *describes exactly the context in which competitive firms operate "by sector", investigated by the traditional partial equilibrium analysis. In these hypotheses the entrepreneur should simply organize the "factors of production" in the least costly way, so as to minimize the price* p_g.

3.2 *The case of constant efficiency*

We assume at first constant efficiency, with the following implications in *System* 2:

a) the total variable costs referred to each period, i.e.: $V_1 = Vg_1(1+r)+Vl_1$; $V_2 = Vg_2(1+r)+Vl_2$;… $V_i = Vg_i(1+r)+Vl_i$;… $V_n=Vg_n(1+r)+Vl_n$; are equal, so that *V* can be adopted as the sole symbol;
b) also the equality $G_1=G_2=…=G_i=…=G_n$ holds, so that *G* can be adopted as the sole symbol.

In these hypotheses, also the capital cost relative to the various periods must be identical, if the price of the product is to coincide for all ages of the machinery[6].

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[5] The two components correspond to the repayment installment (component interest plus component amortization) in the case of debt financing of the purchase of *I* and/or of self-financing (being in the latter case the interest component calculated figuratively); to the leasing installment for capital consumption, in the case of leased machinery.

[6] This "rule" is in general fundamental. It is also especially important "in practice" in the case of leasing, because otherwise the leaseholder would want only the machinery having an age corresponding to the lowest cost.

Therefore, by equating the capital cost between equations 1 and 2, and in general between equations i-1 and i, the following identities result:

$$D_2 = D_1 \ (1 + r) \tag{1}$$
$$D_i = D_{i\text{-}1} \ (1 + r) \tag{2}$$

from which the following relations derive:

$$D_i = D_1 \ (1 + r)^{i\text{-}1} \tag{3}$$
$$\sum_{i=1}^{n} D_i = D_1 \frac{(1+r)^n - 1}{r} \tag{4}$$

From equation (4), owing to the constraint that the sum of annual depreciations must be equal to I, it follows that:

$$D_1 = \frac{Ir}{(1+r)^n - 1} \tag{5}$$

an expression that, having posited I, r and n, allows the possibility to solve for D_1. On the basis of the value D_1 one may then use (3) to compute the value of depreciation for each year from 1 to n. Hence, adding the interest component, it is possible to calculate in any equation (most simply in the first one) the entire annual fixed capital cost F_i, which, being independent from i, can be indicated simply by F. The result is:

$$F = \frac{Ir \ (1 + r)^n}{(1 + r)^n - 1} \tag{6}$$

Substituting F in any equation of *System* 2 for the sum of the two components of the cost of capital, adding the total amount of variable costs V, and dividing by G, the solution for p_g in the case of constant efficiency is finally obtained[7].

In this approach there is a fundamental weakness. If the possibility is considered of introducing new machinery capable of generating output G at a lower annual cost, it may happen that the old machinery which has operated for some years in constant efficiency has to be substituted, since it has become obsolete: therefore n may well be a finite number, of which fixation has to be studied, without excluding *a priori* that it is independent from costs and from the price p_g (c.f. paragraph 4.2.1). But this case must be ruled out, since it is not compatible with the basic Sraffian model we are working within, where a unique technology is being considered. Alternatively, one makes either the highly unrealistic assumption that the capital

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[7] The formula (6) obviously coincides with Sraffa's one relative to constant efficiency, but the analytical process of its derivation differs. This basically follows the traditional method, which Sraffa refers to as typical of "handbooks of commercial arithmetic" (Sraffa, 1960, pp. 64-65). Formally, the final relation for the solution of p_g is:

$$I \frac{r \ (1 + r)^n}{(1 + r)^n - 1} + V = Gp_g$$

"disappears" at the end of the prefixed period n. Otherwise, if constant efficiency persists, it makes no sense to determine n in advance, because as its value rises the cost of capital, F, decreases, tending to its limit at Ir (with D_1, and hence all D_i, tending to zero), while n tends to infinity. This can be immediately verified, if in (6) both the numerator and the denominator of the right-hand side are divided by $(1+r)^n$. As a consequence, in the case of constant efficiency, n must be preset at infinity. The same happens, obviously, in the case of increasing efficiency, or where there is a combination of increasing and constant efficiency.

3.3 *The case of decreasing efficiency*

At a first sight decreasing efficiency is a different matter. In such a case, on one side the cost of capital (i.e., the fixed cost) still decreases when n increases, since the total amortization I is distributed on more and more periods. In particular, the "intensity" of the reduction is of hyperbolic type: i.e., it is very strong when one further year is added to very few years of operation; on the contrary it is weak when there are many previous years of operation. On the other side, the amount of variable costs increases when n increase, often with an exponential type intensity, i.e. positively related to the number of years of operation. Intuitively, there must be a level of n for which total costs reach a minimum. *In other terms, n ceases being a given parameter, and becomes a variable, that in correspondence to a certain level* n^* *generates a minimum in the function associated to n.* In order to investigate this question, I decided to maintain Sraffa's approach, notwithstanding Sraffa holds that also in the case of decreasing efficiency n could be preset, and that it is not possible to find a general solution for F, as the one figured out by equation (6).

In particular, I chose operating on System 2, which, I recall, simply constitutes a transformation of Sraffa's System 1 for the case of a non-basic commodity. As starting point, to represent decreasing efficiency, the assumption $G_1=G_2=...=G_i=...=G_n= G$ is retained, whereas variable costs are hypothesized as increasing or at least semi-increasing over time (i.e., the inequalities $V_1\leq V_2 \leq\ldots\leq V_i \leq\ldots\leq V_n$ hold)[8]. Moreover, we posit as $\Delta V_{i,i\text{-}1}$ (being $\Delta V_{i,i\text{-}1}\geq 0$) the difference of the amount of variable costs between the equation i and the previous one, and as $\Delta V_{i,1}$ the difference of the variable costs between the equation i and the first equation.

In this new situation, let us examine the relations between D_i and D_1, following the same steps as in the case of constant efficiency. The main difference with respect to such a case is that, if we continue to define the capital cost in the traditional way, i.e. as the sum of interest and depreciation, it is not possible to maintain its invariance among equations. In fact, now, if the price p_g has to be kept constant over years, as it

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[8] The increase of variable costs may be due to maintenance and repair costs; to the substitution of pieces of the machinery; or simply to the natural decay of the machinery itself, independently from its intensity of utilization. In any case more intermediate input and/or the application of more labor are necessary, if production has to be kept constant. As regards the part of the increase of variable costs on which to apply the rate of profit r, any parameter h such that $0\leq h\leq 1$ can be chosen. *Moreover, the application of this hypothesis to the whole of the costs referred to intermediate input and labor, differentiating among years, increases the generality of the model: so we will adopt it for the remaining of this work.*

necessary for the model to be consistent, the capital cost has to "accommodate" the variations of the variable costs between subsequent years. In particular, it follows from the assumption about the dynamic of variable costs that the sum of the interest and depreciation components is now decreasing or semi-decreasing. Formally we have the new identities:

$$D_2 = D_1(1+r) - \Delta V_{2,1} \quad (7)$$

$$D_i = D_{i-1}(1+r) - \Delta V_{i,i-1} \quad (8)$$

that imply:

$$D_i = D_1(1+r)^{i-1} - [\Delta V_{2,1}(1+r)^{i-2} + \Delta V_{3,2}(1+r)^{i-3} + \ldots\ldots + \Delta V_{i,i-1}(1+r)^{i-i=0}] \quad (9)$$

The expression (9) indicates that, in the new situation, depreciation at the various ages of a machine can be regarded as a function of the values that it would take in the case of constant efficiency – c.f. (3), diminished by an amount which depends on the variation of the variable costs between all the successive pairs of years up to period *i*, "weighted" by the application of powers of *(*1+*r)* that are higher the closer to the first year of operation of the plant such variation is taken. From (9) the summatory of depreciations up to year *n* can be calculated:

$$\sum_{i=1}^{n} D_i = D_1 \frac{(1+r)^n - 1}{r} - \left[\Delta V_{2,1} \frac{(1+r)^{n-1} - 1}{r} + \Delta V_{3,2} \frac{(1+r)^{n-2} - 1}{r} + \ldots\ldots + \Delta V_{n,n-1} \frac{(1+r) - 1}{r} \right] \quad (10)$$

From (10), constraining the sum of depreciation to be equal to *I*, we can calculate the value of D_1 that satisfies such condition, which is:

$$D_1 = \frac{Ir + \left[\frac{\Delta V_{2,1}}{(1+r)} + \frac{\Delta V_{3,2}}{(1+r)^2} + \ldots + \frac{\Delta V_{n,n-1}}{(1+r)^{n-1}} \right](1+r)^n - \Delta V_{n,1}}{(1+r)^n - 1} \quad (11)$$

In (11) $\Delta V_{n,1}$ derives from the sum $\Delta V_{2,1} + \Delta V_{3,2} + \ldots + \Delta V_{n,n-1}$, which comes about as a step of the transformation of (10). Whereas, the central part of the numerator in the right-hand side has been written in a more complicated form than necessary, since the content of the square brackets will prove useful later on.

Having determined D_1 *through* (11)*, it is then possible to calculate all* D_i *through* (9), *separately from* p_g *and from the other depreciations*, *even in the case of decreasing efficiency*. Correspondingly, the value of the annual capital cost can be determined in a single equation. It is natural to operate on the first equation in *System* 2*,* since: a) we have already calculated D_1; b) in such equation the capital cost is immediately comparable to the investment cost *I*; c) in the first year we do not need considering the variation of variable costs. The following expression is obtained:

$$F_1 = \frac{I\,r(1+r)^n}{(1+r)^n - 1} + \frac{\lfloor \Delta V_{2,1}(1+r)^{n-1} + \Delta V_{3,2}(1+r)^{n-2} + \ldots + \Delta V_{n,n-1}(1+r) \rfloor - \Delta V_{n,1}}{(1+r)^n - 1} \quad (12)$$

We can now try to minimize F_1, by identifying the minimum of (12). More simply, we can find the minimum of (11), since, as is immediately evident from the procedure used in calculating F_1 (D_1 is the only element in the first equation of *System* 2 that varies with n) , the value of n that minimizes (11) also minimizes (12).

To perform such a task, we can calculate the difference $D_1(n) - D_1(n\text{-}1)$ for a generic n , by making use of (11), expressed respectively for the values n and n-1. Recalling that the term $\Delta V_{n,1}$ can be split into addends which correspond exactly to the numerators inside the square brackets in (11), and then elaborating, such difference results equivalent to : $\{Ir\ [(1+r)^{n-1} - (1+r)^n] - \Delta V_{2,1}\ [(1+r)^{n-1} - (1+r)^n - (1+r)^{n-2} + (1+r)^{n-1}] - \ldots\ldots - \Delta V_{i,i-1}\ [(1+r)^{n-1} - (1+r)^n - (1+r)^{n-i} + (1+r)^{n-(i-1)}] - \ldots - \Delta V_{n,n-1}\ [(1+r)^{n-1} - (1+r)^n - (1+r)^{n-n} + (1+r)^{n-(n-1)}]\} / Den$. The denominator *Den*, equal to $[(1+r)^n - 1]\ [(1+r)^{n-1} - 1]$, is not influential for the analysis which follows[9]. The numerator, if the expression $[(1+r)^{n-1} - (1+r)^n] = -r(1+r)^{n-1}$ is factored out, is given by: $-r(1+r)^{n-1}\{Ir - \Delta V_{2,1}\ [1 - 1/(1+r)] - \ldots - \Delta V_{i,i-1}\ [(1 - 1/1+r)^{i-1}] - \ldots - \Delta V_{n,n-1}\ [(1 - 1/(1+r)^{n-1}]\}$. Note that the content of the braces is positive for small values of n. For example, for n=2, its value is $Ir - \Delta V_{2,1}\, r/(1+r)$, where I is generally greater than $\Delta V_2/(1+r)$. On the other side, when n increases, the number of negative terms that "balance" Ir progressively increases, and the component which expresses the variation of variable costs is multiplied by a factor (the content of the square brackets), which clearly increases as well. Therefore, after a certain time-span the content under scrutiny inevitably becomes positive. As a consequence, since the factor $-r(1+r)^{n-1}$ is negative, whereas the denominator, as noted, is positive, $D_1(n) - D_1(n\text{-}1)$ at first is negative, then becomes positive.

The conclusion is that our initial insight (the capital cost shows a *U* shaped dynamic with respect to n, and as a consequence there must be a level of n for which its value reaches a minimum) is verified. To calculate the precise value n^* to which such a minimum corresponds, it is only necessary to equate to zero the numerator of $D_1(n) - D_1(n\text{-}1)$. This simply implies to equate to zero the expression inside braces on which we have elaborate above. In the end, the equation:

$$Ir + \left[\frac{\Delta V_{2,1}}{(1+r)} + \frac{\Delta V_{3,2}}{(1+r)^2} + \ldots\ldots\ldots + \frac{\Delta V_{n,n-1}}{(1+r)^{n-1}} \right] = \Delta V_{n,1} \quad (13)$$

is obtained, *that individuates a unique solution for* n^*. In fact Ir, in the hypotheses made, is positive and constant. Whereas the increase with n of the content of the square brackets in the left-hand side of (13) is slower than the increase of $\Delta V_{n,1}$, since

[9] Except the circumstance that, for r>0, its value is positive.

the latter term, as noted, constitutes the sum of the numerators inside the square brackets, each divided by a number greater than 1, progressively growing with n[10].

However, a second approach to get the solution for n^*, formally simpler and economically more significant than the one just examined, is available. Observe from (9) that for any $n>1$ there will always be a value of D_1 (which will be denoted by D_1^*) such that results $D_n = 0$. This value is given by:

$$D_1^* = \frac{\Delta V_{2,1}}{(1+r)} + \frac{\Delta V_{3,2}}{(1+r)^2} + \dots\dots\dots + \frac{\Delta V_{n,n-1}}{(1+r)^{n-1}} \qquad (14)$$

With D_1^* fixed at such a level, all the D_i preceding D_n ($1 \leq i \leq n-1$) are positives, or at least null (the latter outcome being correspondent to the case $V_{i+1} = V_{i+2} = \dots.. = V_n$). This property is readily demonstrated by substituting D_1 in (9) with the right-hand side of (14).

Let us now examine the sum of the values from D_1 to D_n calculated on the basis of (10), designating its result as S_n, and considering any value for n. Setting in this context D_1 at the value D_1^* which satisfies (14), and all the subsequent D_i up to D_{n-1} at a congruent level according to (9), this sum will yield a certain S_n^*. Values of D_1 lower than D_1^* will produce a sum in which the term D_n (and possibly D_i when i is near n) will be negative, since, again (9) shows, all D_i move in the same direction as D_1 (and therefore, if D_n is equal to zero when $D_1=D_1^*$, it must become negative when $D_1<D_1^*$). Such values are therefore not economically significant, because depreciation cannot be negative. On the other hand, values of D_1 greater than D_1^* yield a sum greater than S_n^*, our term of comparison, with $D_n>0$. With such values for D_1 an amount equal to S_n^* will thus correspond to a period shorter than n: *but its length constitutes an economically unsatisfactory solution*, in that the objective is to reach S_n^* with D_1 (and therefore the cost of capital) at the minimum possible level.

Up to now we have established that for any n the couple D_1^*, S_n^* is optimal from an economic standpoint. On the other hand, it happens that, as n increases, the value D_1^* also increases (or at most remains stationary)[11], and therefore the same happens to S_n^*. It follows that if we start from 1 and let n vary upwards, simultaneously taking for D_1 the value D_1^*, we get sums of annual depreciations S_n^* that minimize D_1 (and hence also F_1), with S_n^* a semi-increasing function of n. So, when S_n^* reaches the value I, the constraint imposed by the replenishment of the initial capital value is satisfied in an optimal way, i.e. at the minimum annual capital cost: n

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[10] Note that, if r=0, the equation (13) is indefinite. However, a solution can be worked out, by positing r=0 in the relation (9), and then using the expression so obtained to calculate D_i in the passages that follow. The result of the process is: $I = \Delta V_{2,1} + 2\Delta V_{3,2} + \dots + (n-2)\Delta V_{n-1,n-2} + (n-1)\Delta V_{n,n-1}$.

[11] Most simply, if in (14) we consider the values D_1^* associated to periods n and $n+1$, calling them respectively by $D_1^*(n)$ and $D_1^*(n+1)$, we have:

$$D_1^*(n+1) - D_1^*(n) = \Delta V_{n+1,n} / (1+r)^n$$

Since it has been previously assumed that $\Delta V_{i,i-1}$, and therefore $\Delta V_{n+1,n}$, is ≥ 0, the same conclusion follows for $D_1^*(n+1) - D_1^*(n)$.

corresponds therefore to n^*. Of course, the same result is obtained by comparing, during the process of increase of n, the dynamic of D_1^* with the dynamic of D_1 resulting from (11) – let us call it $D_1°$ –, i.e. with the value of D_1 that guarantees the full amortization of I. In other terms, at the value of n for which $D_1° = D_1^*$ fixed costs, and therefore in our hypotheses total costs, are minimized. *In conclusion, both the "twin" procedures are capable of individuating the "second approach" solution for* n^* *we are looking for.*

3.4 ***The formalization of the solution in the second approach***

Four formal methods to calculate n^* within this second approach will be indicated. The first one has just been outlined: the solution must derive from the equality $D_1° = D_1^*$, i.e. from the equality between the right-hand side of (11) and of (14). The elaboration of the result is made easy by the circumstance that the right hand side of (14) corresponds exactly to the content of the square brackets in (11). *In the end we obtain an expression coinciding with* (13)*, which proves the validity of the second approach*. Of course, an analogous result is derived from the application of the "twin procedure", which implies the substitution of the right hand-side of (14) for D_1 in (10), where in the left hand-side the sum of depreciations is posited to I.

The second method derives from the observation that in the first equation of *System* 2 variable costs are at their "basic" value, and the cost of capital F_1 corresponds to $Ir+D_1°$, i.e. to the amount of interest resulting from the application of r to the initial value of the capital plus the "replenishment amortization" attributable to the first year - that of equation (11), which above has been called $D_1°$. On the other hand, in the last equation (the n^{th}) both these magnitudes are equal to zero, since the capital has been completely "repaid" and at the same time amortization is null. Therefore, since the same price p_g applies both to the first and the last equation, the equality $Ir + D_1° = \Delta V_{n,1}$ must hold. *Substituting* F_1*, i.e. the right-hand side of* (12)*, in place of* $Ir+D_1°$*, and then elaborating, an identical expression to* (13) *is again obtained.*

The third method constitutes a simple variation of the second one. If in the n^{th} equation amortization and interest are null, the capital cost, as measured in it, must be null as well, i.e. the equality $F_n = 0$ holds. Since $F_n = F_1 - \Delta V_{n,1}$, the final consequence is $F_1 = \Delta V_{n,1}$, which corresponds exactly to the second method. This method has been reported since, with respect to the two previous ones, the emphasis is shifted from the first to the last equation of *System* 2, and the circumstance that the last equation is composed only by variable costs is more directly pointed out.

Both these characteristics are also evident in the fourth method. This is based on the observation that, in addition to the circumstance that in the n^{th} equation the capital cost is zero, and so the price p_g depends entirely on the variable costs, it is also true that in the equation before the n^{th} (the $n^{th\text{-}1}$) the capital cost becomes positive, but its variation above zero can be detected on the basis of the variation of the variable costs, with respect to their amount in the last equation. The same happens for the $n^{th\text{-}2}$ equation and for the other equations up to the first one, *so that the detection of the* *solution for n^* may be carried out exclusively through the variations of the variable costs.* To illustrate, consider in the first place that, in the $n^{th\text{-}1}$ equation, the "room" for the capital cost, which in the last equation was null, becomes equal to $\Delta V_{n,n\text{-}1}$; in the $n^{th\text{-}2}$ this "room" becomes equal to $\Delta V_{n,n\text{-}1} + \Delta V_{n\text{-}1,n\text{-}2}$; and so on, up to the first equation, in which the "room" is equal to: $\Delta V_{n,n\text{-}1} + \Delta V_{n\text{-}1,n\text{-}2} + \ldots + \Delta V_{2,1}$. Moreover:

a) the sum of the present values of the various "rooms" for the capital, referring at the beginning of the first period of the life of capital, must be equal to I;
b) such present values are obtainable by dividing the "room" in the $n^{th\text{-}1}$ equation, that corresponds to $\Delta V_{n,n\text{-}1}$, by $1/(1+r)^{n\text{-}1}$; by dividing the "room" in the $n^{th\text{-}2}$ equation, that corresponds to $\Delta V_{n,n\text{-}1} + \Delta V_{n\text{-}1,n\text{-}2}$, by $1/(1+r)^{n\text{-}2}$; and so on, up to the first equation, where the relevant "room", $\Delta V_{n,n\text{-}1} + \Delta V_{n\text{-}1,n\text{-}2} + \ldots + \Delta V_{2,1}$, is to be divided by $1/(1+r)$;
c) in the summatory deriving by the indications of the two previous points the various $\Delta V_{i,i\text{-}1}$ (with $1 \le i \le n$) can be factored out, and the sum which multiplies each of them, constituting a geometric series, may be expressed in a compact form;

so that in the end the following expression results:

$$I = \Delta V_{n,n-1} \frac{\left[(1+r)^{n-1}-1\right]}{r\left(1+r\right)^{n\text{-}1}} + \Delta V_{n-1,n-2} \frac{\left[(1+r)^{n-2}-1\right]}{r\left(1+r\right)^{n\text{-}2}} + \ldots\ldots\ldots + \Delta V_{2,1} \frac{\left[(1+r)-1\right]}{r\left(1+r\right)} \tag{15}$$

which again, after little elaboration, can be transformed exactly into (13). Since the above expression indicates the solution for the optimal life of the plant by evidencing its initial value, but not at all its annual costs, which are only "mirrored" by the variations of the variable costs, in the rest of this work I will refer to (15) as "the mirror equation".

After determining n^*, it is easy to proceed to the calculation of p_g as the final step. With the first two methods illustrated above it is "natural" to carry out this operation in the first equation: first by calculating F_1 – through (12); then by adding V_1 to F_1; and finally by dividing the result by G_1. *The simultaneous solving of System* 2 *is so avoided, as in the case of constant efficiency*. The same procedure may be followed, of course, for the third and the fourth method for the solution of n^*. But for them, and especially regarding the latter, it becomes natural to solve for p_g in the last equation, by simply dividing its left-hand side, i.e. $\Delta V_{n,1} + V_1 = V_n$, by G_n (G_n is equal to G in our previous hypotheses; but the symbol G_n has been adopted, in order to be quoted later in the resulting formula). In formal terms we have:

$$p_g = \frac{V_n}{G_n} \tag{16}$$

Since p_g constitutes the price of a commodity produced by making use of a capital good, but in equation (16) nothing about such good appears, I will refer to it as "the shadow equation".

The four methods share the characteristic to highlight the solution for n^* *"from the left-hand side" of it, i.e. by concentrating the analysis on the dynamic of economic values from year* 1 *to* n^*. *The approach is therefore the opposite of the "truncation method"- see paragraph* 2.2, *which concentrates the analysis on the economic values which are "on the right-hand side" of* n^*[12]. This may explain why the most important feature of the optimal solution, i.e. that the value of capital should be equal to zero in the left hand-side of the last equation of *Systems* 1 and the capital cost must show an analogous characteristic in the last equation of *System* 2, has not been displayed by that method.

3.5 *Numerical examples*

An illustrative example can be helpful here. We set the three variables beyond the unknown n^*, which appear in all four methods of solution and in their common technical transformation, i.e. in equation (13), at the following levels: $I = 100$; $r = 0{,}10$; $\Delta V_{n,n\text{-}1} = 3$ (with the first variation of variable costs registered between years 2 and 1, as the calculus between years 1 and the non-operative year zero is not possible). On these assumptions, the dynamic of variable costs is described by the expression $V_n = 3(n\text{-}1)$, and for years 1 through 15 the four methods yield the values reported in *Table* 1 and represented in *Figure* 1, which follow.

Table 1[13]

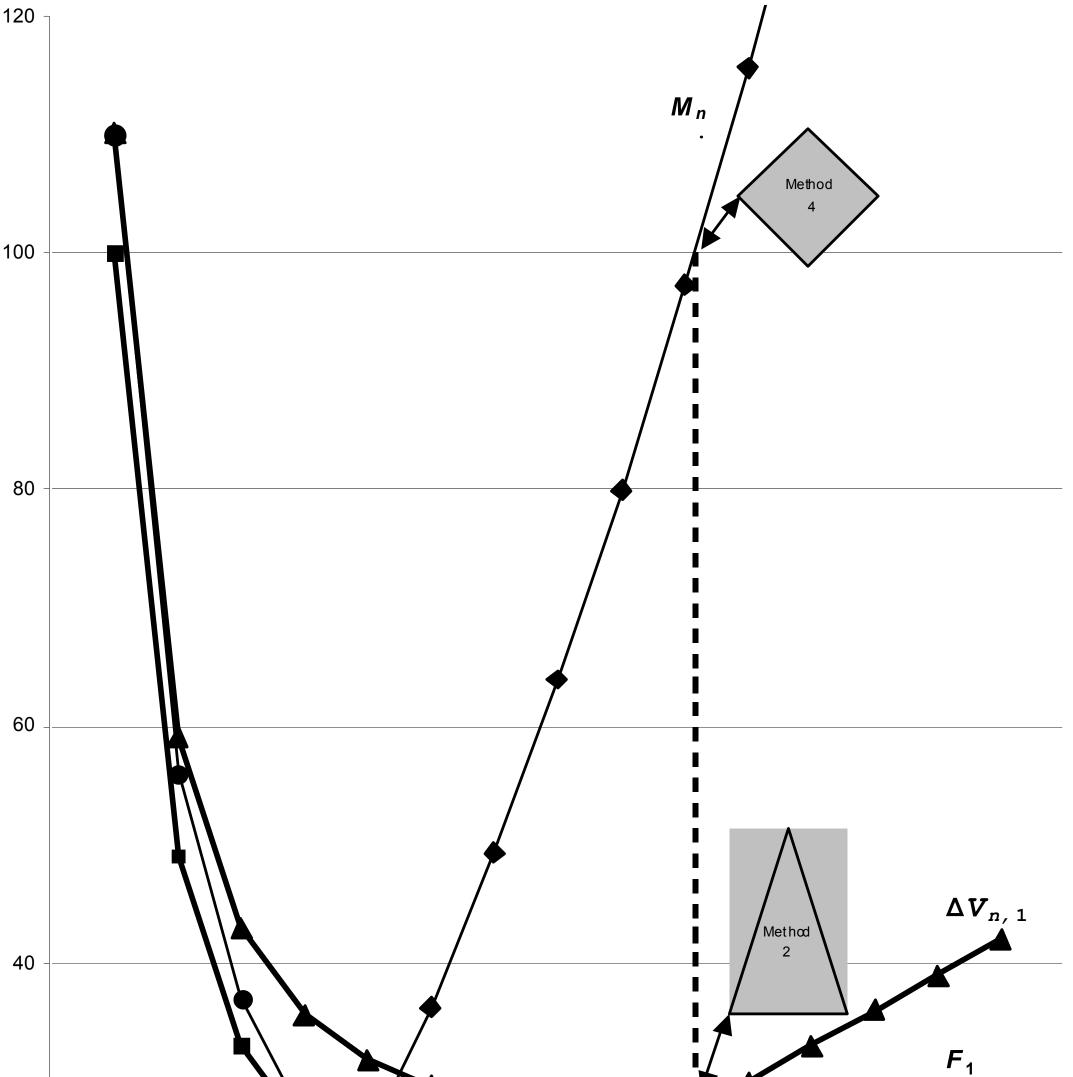
Example of the four methods of solution for n^* *(second approach)*

	n	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
Method 1	$D_1°$	100	49,05	33,02	25,69	21,81	19,63	18,41	17,76	17,48	17,45	17,59	17,84	18,17	18,56	18,98
	D_1^*	0	2,73	5,21	7,46	9,51	11,37	13,07	14,61	16	17,28	18,43	19,49	20,44	21,31	22,1
Method 2	F_1	110	59,05	43,02	35,69	31,81	29,63	28,41	27,76	27,48	27,45	27,59	27,84	28,17	28,56	28,98
	$\Delta V_{n,1}$	0	3	6	9	12	15	18	21	24	27	30	33	36	39	42
Method 3	F_n	110	56,05	37,02	26,69	19,81	14,63	10,41	6,76	3,48	0,45	-2,41	-5,16	-7,83	-10,44	-13,02
Method 4	M_n	0	2,73	7,93	15,39	24,9	36,28	49,34	63,95	79,95	97,23	115,7	135,15	155,59	176,9	199

[12] In the first approach developed in this paragraph, that is based on mathematical calculus to solve for the minimum of a relation, both the right and the left hand side of n^* are involved, but only to determine the sign of the slope, without "insights" on the dynamics of the relevant economic magnitudes (similarly in particular to Kurtz and Salvadori's approach – c.f. paragraph 2.2).

[13] The symbol M_n stands for the right-hand side of the mirror equation.

Figure 1
*Graphical representation of the four methods of solution for n**



It can be observed that the optimal life n^* comes about after the beginning of the tenth and before the eleventh year[14]. Of course also the relation (13) is satisfied inside that period. This can be verified by observing that *Ir* is a constant value equal to 10, and that the sum $10+D_1^*$, with D_1^* representing the brackets in the left-hand side of (13) - compare (13) and (14) -, reaches $\Delta V_{n,1}$, i.e. the right-hand side of (13), after the beginning of the tenth year of life of the plant. Observe also from *Table* 1 that, after the tenth year, negative values (and therefore unacceptable) for the cost of the capital F_n appear, whereas the value D_1° reaches its minimum in the same interval: both these circumstances constitute further confirmations of the validity of our methods. [15]

Once n^* is determined, through the relation (9), all the D_i consistent with D_1°, and their progressive sum as well, may be calculated. Moreover, in order to make a comparison of the results, the same operation can be carried out for whatever *n*, after determining D_1 through (11). Specifically, in *Table* 2, beyond the tenth year, corresponding to the optimal period n^*, two further *n* have been selected:

1) the fourteenth year, which corresponds to the case n_1 in paragraph 2. In other words: a) negative values for the capital would appear in *System* 1 (in particular starting from the seventh year on, since for such periods the sum of D_i from 1 to *i* exceeds the initial value of the capital); b) the capital cost at $n_1 = 14$ overcomes that of the optimal year. The difference is approximately one point, as it can be seen from the table by comparing the values of this case to those of n=10, relatively to i=1 (being the difference of amortization in the first year equal to that of the capital cost, since the interest component is *Ir* in both cases);
2) the seventh year, which corresponds to the case n_0 in paragraph 2. In other words, there are not negative values for the capital but the capital cost exceeds that of the optimal year. The difference is approximately one point, similarly to the fourteenth year choice (compare the relevant values for i=1 in *Table* 2 again).

It can be noted that:

a) the profile of amortization, in each case and not only for the optimal choice, is

[14] Once the period inside which the four methods indicate the placement of the solution has been identified, this may be divided into sub-periods, in such a way that the fraction of the last year for which operating is economically advantageous can be precisely determined. In our case, after: a) dividing the tenth year by 100; b) attributing to each micro-period 0,03 as increase of variable costs and $1/(1+0{,}10)^{10}$ as "discount factor"; this fraction results 0,15, so that the optimal duration is 10,15. Following the same procedure, but adopting the equation in footnote 10, in the case r=0 the result is 8,666.

[15] *Some commentators of this work have expressed the view that the above methods of solution (in particular method* 2) *reflect the marginalist equilibrium between average cost and marginal cost. Such a view is totally wrong. Specifically, as regards method* 2, *on the one hand* F_1/G_1 *does not represent an average cost. In fact, its variation is referred to increments of n* (*and not to increments of the product in the first year,* G_1 *, which is given!). On the other hand,* $\Delta V_{n,1}$ *does not represent a marginal cost, even in the case that the variation of production is considered in function of n (for which a variability can be calculated, differently than inside single periods*). *In fact* $\Delta V_{n,1}$ *represents the "total"* (*i.e. between periods n and* 1), *not the "marginal" variation* (*i.e. between periods n and n*-1) *of the variable costs when n increases, and so any attribution of this variation to single periods is meaningless.*

Table 2

Amortization in each year, and total up to i, with n posited at 10,14*, and* 7 *years*

	i	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
n=10	$\boldsymbol{D_i}$	17,45	16,20	14,81	13,30	11,63	9,79	7,77	5,54	3,10	0,41	=	=	=	=	=
	$\sum_{i=1}^{i} D_i$	17,45	33,65	48,46	61,76	73,38	83,17	90,94	96,48	99,58	99,99	=	=	=	=	=
n=14	$\boldsymbol{D_i}$	18,56	17,42	16,16	14,77	13,25	11,58	9,73	7,71	5,48	3,03	0,33	-2,64	-5,9	-9,49	=
	$\sum_{i=1}^{i} D_i$	18,56	35,98	52,13	66,91	80,16	91,73	101,5	109,2	114,6	117,7	118,0	115,4	109,5	99,97	=
n=7	$\boldsymbol{D_i}$	18,41	17,25	15,98	14,57	13,03	11,33	9,47	=	=	=	=	=	=	=	=
	$\sum_{i=1}^{i} D_i$	18,41	35,66	51,64	66,21	79,24	90,58	100,1	=	=	=	=	=	=	=	=

of hyperbolic type[16];

b) as predicted in paragraph 2.1, in the fourteen year choice the capital value becomes negative before the optimal year, which means that it touches "point zero" twice[17];

c) if truncation is carried out by eliminating the equations in which negative values for capital appear, the result is the passage from the second to the third case, which entails an annual capital cost which is still higher than the optimal choice (in our example above, more or less by the same amount as in the second case!).

Finally, it is evident from (13) that it is possible to express the dynamic of variable costs as a mathematical function, and then it is easy to pass from discrete-time to continuous-time analysis. For example, in the hypothesis that $\Delta V_{n,n\text{-}1}$, in the discrete-time analysis, is a simple linear function of n of the type $\Delta V_{n,n\text{-}1} = a(n - 1)$, in which a is a fixed parameter, representing a constant variation of the variable costs between time units, concentrated at the beginning of each time unit starting from the

[16] This demonstrates that the fiscal practice of regarding a constant annual amount of depreciation as the "correct" procedure, and accelerated amortization as a benefit, must be reversed: accelerated depreciation is the norm, whereas constant depreciation is surely a penalty. Of course the correct profile of accelerated depreciation must be determined. This work shows how to perform such task scientifically.

[17] Formally, this implies that *System* 1 in paragraph 2.1 must be reset. The value of capital on the output side is equal to zero not only in the last equation, but also in an equation "internal" to the system. In the equations which follow the latter the capital value becomes negative, with absolute numerical determinations increasing at first (when amortization is still positive), and then declining (when amortization becomes negative, i.e in the twelfth year – c.f. *Table* 2 in the row corresponding to D_i) up to the zero value again in the last equation.

second unit (so that there is a perfect correspondence with the previous approach – in which the time-unit was one year)[18], the relation (13) in the continuous case can be expressed as follows[19]:

$$I\left(e^{r}-1\right)+\frac{a}{r}\left(1-e^{-nr}\right)=an \tag{17}$$

The relation (17) can be used to calculate the optimal n^* for any parameter, in particular for any r>0. The solution is still "computational", but the calculations are simple and transparent[20].

Another example may be constituted by an exponential function, of the type $\Delta V_{n,n\text{-}1} = A[(1+b)^{n\text{-}1}-1]$, where A and b are positive parameters, and the subtraction of the unity has the function to set at zero the variation of variable costs inside the

[18] Eventual repair and maintenance expenses in the first instants of operation of the plant, and/or the expenses to start the plant, are absorbed by the value V_1 .

[19] The transformation of (13), after introducing the linear function $\Delta V_{n,n\text{-}1} = a(n\text{-}1)$, but still maintaining the discrete-time analysis, implies:

$$Ir+\left[\frac{a}{(1+r)}+\frac{a}{(1+r)^{2}}+...+\frac{a}{(1+r)^{n-2}}+\frac{a}{(1+r)^{n-2}}\right]=a\,(n-1) \tag{13*}$$

in which the relevant magnitudes (r, a) are referred to the time-unit of (13). Passing to the continuous-time analysis, the first term on the left-hand side of (17) is obtained by transforming the analogous term of (13*) as follows: $Ir/x\ [1+(1+r/x)+(1+r/x)^2+...+(1+r/x)^{x-1}]$, where x constitutes a fraction of the relation (13*) time-unit, and then by: a) calcutating the sum of the geometric series inside the brackets; b) eliminating through simplification r/x ; c) tending to infinity x . Following the above economic logic, the second term on the left-hand side of (13*) can be changed into the expression:

$$-\left[\frac{a/x}{(1+r/x)}+\frac{a/x}{(1+r/x)^{2}}+\frac{a/x}{(1+r/x)^{3}}+.....+\frac{a/x}{(1+r/x)^{nx}}\right]$$

that, after: a) factoring out $a/x\ /(1+r/x)$; b) calculating the summatory inside the square brackets relatively to n for a given x; c) simplifying and tending x to infinity; generates the analogous term on the left-hand side of (17). Finally, the term on the right-hand side of (17) is obtained by transforming the analogous term in (13*) as follows: $(a/x)(nx-1) = an - a/x$, and then by tending x to infinity.

[20] In the same hypotheses as in the discrete case (I=100; a=3; r=0,10) the solution n^* of (17) is approximately 9,72, i.e. some fraction less than in the discrete case (where it is 10,15 , as indicated in footnote 14). The economic reason of the difference is that, with the passage to the continuous time analysis, a small switch of the barycentre of the increase of variable costs, which is on average anticipated, comes about.

In the assumption that different levels of r do not affect a (i.e that the parameter h of footnote 8 is zero – c. f. paragraph 4.1.4 for the case h>0), and in the same hypotheses as above for I and a : a) with r at 0,15 and 0,20 , the solution n^*, as calculated in (17), is respectively 10,73 and 11,92; b) on the other hand, with r at 0,05 and 0,001 (i.e., approximately at zero) , the results are respectively 8,88 and 8,17. Therefore it can be observed that 20 points of variation of r starting from near zero bring about a significant increase of n^*, i.e. almost 4 years.

For the case r=0, the expression (17) becomes indefinite, but it is possible to resort to the expression outlined in footnote 10. This, in the continuous case, becomes $I = an^2/2$, from which the general solution $n^* = \pm\sqrt{2\,I/a}$ is derivable, where obviously only the positive determination can be accepted. With I=100 and a=3 the result is 8.16: the approximation r=0,001 in (17) is therefore very good. The difference with respect to 8,666, the result in the discrete case (c.f. footnote 14), can be explained as above for r=0,10.

first period, in analogy with the linear case. The passages to the continuous-time approach generate the following result[21]:

$$I(e^{r}-1) \;+\; \frac{Ab}{r-b}\left[1-e^{(b-r)n}\right] \;=\; A(e^{bn}-1) \qquad (18)$$

Therefore, also in the exponential case an indirect (but easy to work with) relation for the solution for n^* is obtained[22].

4. Extension of the results

4.1 *Extensions within Sraffa's framework*

4.1.1 *Increasing efficiency.* The model developed in section 3 is compatible with increasing efficiency. Of course, increasing efficiency must regard limited periods. In fact, in the years of increasing efficiency, i.e. when ΔV_i is negative, in the period i-1 the variable cost is greater than in period i, so that the "room" for the capital cost in period i-1 is negative. Given that the interest component is surely positive, this means that amortization is negative. So, the full amortization of the capital good can never be reached, since a sum of negative values can never get equal to the positive value I. From an economic standpoint, the optimal n^* must be posited at the infinite, exactly as it happens in the case of constant efficiency.

On the other hand, when increasing efficiency regards limited periods, there is full compatibility with a definite solution for n^*. In particular, it is worth outlining three main possibilities:

a) a U-shaped configuration of the curve of efficiency, with increasing efficiency prevailing at first and decreasing efficiency prevailing later[23];
b) an efficiency curve with "peaks" and "valleys", in a trend of decreasing efficiency;

[21] The first term on the left-hand side of (18) is determined exactly as in the linear case. The second term on the same side is obtained by the following expression, derived from the application of the exponential function to the analogous term in (13):

$$A\left[\frac{b/x(1+b/x)^0}{(1+r/x)}+\frac{b/x\,(1+b/x)^1}{(1+r/x)^2}+\frac{b/x\,(1+b/x)^2}{(1+r/x)^3}+\ldots+\frac{b/x\,(1+b/x)^{xn-1}}{(1+r/x)^{xn}}\right]$$

that, after: a) factoring out $b/x/(1+r/x)$; b) calculating the summatory inside the square brackets relatively to n for a given x; c) simplifying and tending x to infinity; generates:

1) Abn, for $b=r$;
2) the second term on the left-hand side of (18), for b respectively greater or smaller than r.

Finally, the term on the right-hand side is obtained by transforming the function which expresses the dynamic of variable costs, i.e. $A\,[(1+b)^{n-1}-1]$, into $A\,[(1+b/x)^{nx}/(1+b/x)-1]$, and then tending x to infinity.

[22] With I=100; A=14 ; b=0,1, it is obtained: for r=0,10 , n^*=10,19 (very near to the linear case with r=0,10); for r=0,15 , n^*=10,99; for r=0,20 , n^*=11,87 ; for r=0,05, n^*=9,46; for r=0,001 , n^*=8,85. The difference between the maximum and the minimum is about 3 years, i.e. is relevant but smaller than for the example of the linear case.

[23] A configuration of such type may be a quadratic function, for example $V_n = -a(n\text{-}1) + b(n\text{-}1)^2$, which, being the derivative $-a + 2bn$, if $a > 2b$, decreases for low values of n but increases for greater values. The negative slope may be associated to intensive "breaking in" expenses in the initial periods of functioning of the plant.

c) a configuration of efficiency similar to a combination of the two previous ones, but with a great peak of current outlays after some years (due, for example, to substitution of an important piece of machinery, or to its complete revision), followed by "a valley" of increasing efficiency. This case is interesting, because in the year (or the years) of the "big peak" the cost of capital may become negative, but it is convenient to use the plant for further years, to take advantage, in the following years, of declining outlays[24].

4.1.2 *Temporal variability of output.* The case of variability of the output G during the life of the plant can be treated along the same schematic as the variability of variable costs, i.e. through the search of the solution starting from the direct analysis of the dynamic of capital cost. This should mean solving for n^* independently from the price p_g, which is formally calculated *afterwards*, as shown in paragraph 3.3 and 3.4. As a matter of fact, however, p_g is implicitly determined in the process of searching for n^*, since, whatever the solving method adopted, for each "trial" relative to n the two variables which appear in (16), i.e. V_n and G_n, enter the picture. We can make this circumstance explicit, obtaining the advantage of strongly simplifying the analysis, since, in comparing the various years, it becomes possible to refer not only to different quantities of output, *but also to different values of such output*. This procedure matches especially well with the fourth method of the second approach (the one, developed in paragraph 3.4, based on the mirror equation), since its initial emphasis is on the last equation, where the price is determined, and then the precedent equations come under consideration, with the calculation of "rooms" for amortization and interest on residual capital.

In the case under examination the room for the capital cost is still zero in the n^{th} equation. It is positive in the $n^{th\text{-}1}$ equation, where it is composed by the two addends: $\Delta V_{n,n\text{-}1} + (-\Delta G_{n,n\text{-}1})\, V_n / G_n$. In this expression $\Delta V_{n,n\text{-}1}$ is the specific room created by the variation of variable costs; V_n / G_n represents, for each "trial" for n, the price p_g; whereas $\Delta G_{n,n\text{-}1}$ represents the variation of output between year n and year n-1. Since $\Delta G_{n,n\text{-}1}$ is ordinarily negative (being in hypothesis $\Delta G_{i,i\text{-}1} \leq 0$ for any i), whereas a value of the output greater in n-1 than in n surely generates a positive room for the capital cost, the negative sign is attributed to its amount. In the $n^{th\text{-}2}$ equation the appropriate room is: $(\Delta V_{n,n\text{-}1}+\Delta V_{n\text{-}1,n\text{-}2}) + [-(\Delta G_{n,n\text{-}1}+\Delta G_{n\text{-}1,n\text{-}2})]\, V_n / G_n$. Finally, in the first equation, the room is: $(\Delta V_{n,n\text{-}1}+\Delta V_{n\text{-}1,n\text{-}2}+\ldots+\Delta V_{2,1}) + [-(\Delta G_{n,n\text{-}1}+\Delta G_{n\text{-}1,n\text{-}2}+\ldots+\Delta G_{2,1})]\, V_n/G_n$.

After going through the three steps a), b), c) of method 4, we obtain:

$$I=\left[\Delta V_{n,n-1}-\Delta G_{n,n-1}\frac{V_n}{G_n}\right]\frac{(1+r)^{n-1}-1}{r(1+r)^{n-1}}+\left[\Delta V_{n-1,n-2}-\Delta G_{n-1,n-2}\frac{V_n}{G_n}\right]\frac{(1+r)^{n-2}-1}{r(1+r)^{n-2}}+\ldots$$
$$+\left[\Delta V_{2,1}-\Delta G_{2,1}\frac{V_n}{G_n}\right]\frac{(1+r)-1}{r(1+r)} \qquad (19)$$

that constitutes the equivalent of (15), and that can be transformed into:

[24] Of course in this hypothesis during the year(s) of peak outlays the capital cost may be negative, but the residual value of the plant (initial value minus the total of previous amounts of amortization) is still positive.

$$Ir+\left[\frac{\Delta V_{2,1}-\Delta G_{2,1}\dfrac{V_n}{G_n}}{(1+r)}+....+\frac{\Delta V_{n,n-1}-\Delta G_{n,n-1}\dfrac{V_n}{G_n}}{(1+r)^{n-1}}\right]=\Delta V_{n,1}-\frac{V_n}{G_n}(G_n-G_1) \tag{20}$$

The relation (20) is equivalent to the "basic" relation (13)[25], and allows to calculate n^* easily, also for the case of variability of output. After solving for n^*, the price p_g may be simply evidenced. In fact, its value is already implicitly calculated in (19) and in (20) – where, as noted, it corresponds to V_n/G_n. Finally, for the case of increasing efficiency expressed through increases of output, analogous considerations to those developed in paragraph 4.1.1 apply.

4.1.3 *Basic production.* Now let us examine the case that G is a basic product, so that the value of p_g interacts with that of its costs, both variable and fixed (the latter through the simultaneous determination of p_g and of the value I).

The analysis will be carried out in particular by referring to relation (19), that, since "condensates" the mirror and the shadow equations, shows an economic meaning clearer than relation (20). In order to return to the magnitudes of *System* 1, in such relation I must be replaced by $M_1 p_{m1}$; $\Delta V_{n,n-1}$ must be replaced by $[(A_{gn}- A_{g,n-1})p_a + ...+ (K_{gn}- K_{g,n-1})\, p_k]\,(1 + r) + (L_{gn} - L_{g,n-1})\, w$; $\Delta V_{n-1,n-2}$ must be replaced by: $[(A_{g,n-1} - A_{g,n-2})p_a +...+(K_{g,n-1} - K_{g,n-2})p_k]\,(1 + r)+(L_{g,n-1} - L_{g,n-2})w$. The same procedure must be applied to the other operating years, with respect to the variable costs of the preceding year. Specifically, in the second operating year, $\Delta V_{2,1}$ must be substituted by: $[(A_{g2} -A_{g1})p_a+...+ (K_{g2}- K_{g1})\, p_k\,](1+r)+ (L_{g2} -L_{g1})w$[26]. Finally, V_n must be replaced by $(A_{gn}p_a+...+K_{gn}p_k)\,(1+r) + L_{gn}w$; whereas $\Delta G_{2,1}$, $\Delta G_{3,2}$; $\Delta G_{n,n-1}$, G_n, are not to be replaced.

The resulting expression brings about the possibility to regard the price $p_g = [(A_{gn}p_a+...+K_{gn}p_k)\,(1+r) + L_{gn}w]/G_n$ as an implicit function of all other prices p_{m1}, p_a ,….. p_k , considered as independent variables. The same can be done for each production process, so that a number of relations equal to the number of commodities is obtained, each one expressing implicitly the price of a single commodity in terms of prices of all other commodities. These relations show three characteristics:

a) they are defined in the positive quadrant of the orthogonal axes;
b) the derivative of the "dependent" price with respect to each "independent" price is positive;
c) the functions relative to whatever couple of prices (as an example, p_a as a function of p_b and p_b as a function of p_a), calculated while keeping fixed at a certain level the remaining prices, are convergent;

[25] In particular (20) becomes equal to (13) when all the $\Delta G_{i,i-1}$ are null, and therefore also $G_n - G_1$ is equal to zero.

[26] Note, however, that the hypotheses developed in the preceding paragraphs are more general than Sraffa'ones, as regards the possible levels of "anticipation" of variable costs (c.f. footnote 8).

so we are sure that a vector of positive simultaneous solutions for all prices exists[27]. After determining the prices of commodities, it is also possible, relative to capital goods, to calculate, for each operating year, the amount of amortization (and the residual value as well) by making use of the relations that describe the productive processes from year 1 to year n^*-1[28]. Of course, whatever the structure of the general solution for prices is, the fundamental circumstance remains that both the price of capital goods and amortization tend to be zero in the last period of functioning.

4.1.4 *Different levels of the rate of profit.* In a context of general equilibrium, different levels of r imply different structures of relative prices of all commodities. Therefore, the optimal life of each machine may vary in the first place because the relative values of the capital good I and of variable costs ($\Delta V_{n,n\text{-}1}$, $\Delta V_{n\text{-}1,n\text{-}2}$,….. $\Delta V_{2,1}$, and V_n) change.

But it is important also to consider the "basic" relation between n^* and r, i.e. their link in the hypothesis that such a change does not occur, because the value of the capital goods and variable costs change in the same proportion – case in which, as is apparent from equations (19) and (20), the relation between n^* and r is not altered.

In such a circumstance, let us first explore the case that no part of the variable costs is anticipated (corresponding to h=0 in footnote 8), so that different levels of r affect directly neither $\Delta V_{n,n\text{-}1}$, $\Delta V_{n\text{-}1,n\text{-}2}$, …, $\Delta V_{2,1}$, nor V_1 and V_n, and as a consequence do not affect the content of the square brackets which constitutes the first of the two components of each addend in the right-hand side of (19) − again, the only solving equation for n^* that will be considered. On the other hand, greater levels of r, in (19), are associated to smaller values of the second component of these addends (i.e. the factor which multiplies each couple of brackets), since its derivative with respect to r bears a negative sign for any positive n.

Therefore, if $r' > r$ is posited, it is possible to calculate the right-hand side of (19) for both rates of profit and the amount of the positive gap between the two results. The only possibility to "fill the gap", and so to restore full amortisation of the initial outlay for the investment, is increasing the number of periods which the plant functions. The "sensitivity" of the optimal life of the capital with regard to r is in general relevant, and depends also on the temporal profile of the variation of the variable costs[29].

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[27] More precisely, negative prices are excluded, if $r<R$, i.e. if the rate of profit is posited to a level inferior to its possible maximum value. This maximum value can be determined with certitude empirically, by letting r to assume levels progressively higher starting from zero up to the position for which the price of one commodity, and therefore of all commodities, approaches infinity. Research is open for a determination of R from a theoretical point of view. On this Sraffa's method must be excluded, since it is based on the hypothesis that the quantities of input and output which form the system are pre-defined independently from r, which, being n^* variable with r, is not correct.
Another circumstance to be verified is that the pre-fixed level of output of each sector, regarded as a productive input for the other sectors, is at least sufficient. The quantities of intermediate input are not in fact pre-fixed, differently than in Sraffa's system, since they depend on the optimal life of capital in each sector, which in our framework constitutes an unknown. On this note that the relation between each price and the other prices, as presented above, is defined in every point in correspondence with the optimal n^*, which therefore is determined together with the general solution for prices.

[28] Therefore such operation is not simultaneous with the determination of the price of currently produced commodities.

[29] This has been verified in particular in the examples of footnotes 20 and 21.

In the case that a part or the totality of the variable costs is anticipated ($0<h\leq1$), a greater r brings about a positive variation of $\Delta V_{n,n\text{-}1}$, $\Delta V_{n\text{-}1,n\text{-}2}$,…$\Delta V_{2,1}$, and of V_1 and V_n as well, more or less in the same proportion, and as a consequence the first component of the addends in the right-hand side of (19) increases. Therefore, it is not guaranteed that the above gap is negative. In particular, higher values of h and of the ratio between variable costs and the value of the capital foster the possibility that the contrary is positive.

4.1.5 *Implications for Sraffa's framework.* The main consequences for Sraffa's system can be outlined as follows:

a) the solution for prices of commodities, when fixed capital is considered, can be determined without resorting to the analytical structure which is proper to joint products. This possibility has already been considered (Roncaglia, 1971). However, in this work emerges as a natural "by-product", in the context of a representation of economic activity based on simple and "traditional" concepts;
b) the reduction of each price in terms of "dated labor" becomes possible even in the presence of fixed capital. In fact, in the last equation of the system, only intermediate inputs appear, so the reduction may be carried out as in a scheme without fixed capital. Moreover, such an operation is possible also for other operating periods, by attributing to each one the quota of labor embodied in the new plant, according to the ratio between the amortization of the period and the initial value of the plant, and then by "discounting" the result at the appropriate rate. *This outcome, together with the circumstance that the solutions of the system are obtained in a context of minimization of costs, leads to the conclusion that Sraffa's prices must be ultimately considered as minimizing the actual value of "embodied" dated labor*;
c) since the quantities relative to the commodities of the standard system, when there is fixed capital, are not independent from the rate of profit, it is impossible to construct a law of distribution between profits and wages before knowing the value of unknown prices. However, such distribution can still be calculated at a separate stage of the analysis, precisely after the determination of prices (i.e. the minimization process does not entail an endogenous determination of the rate of profit, as in the marginalist theory). In particular, by making use of the reduction to dated quantities of labor, it is possible to calculate, for every "final good" (destined for consumption or for investment), the "content" in terms of wages and gross profits (inclusive of amortization), that corresponds to specific quotas of total earned wages and of total gross profits.

4.2 ***Beyonds Sraffa's framework***

4.2.1 *Technical progress.* This subject is treated very briefly, mainly to point out an interesting result, that partially restores Sraffa's analysis on fixed capital, in particular relative to the case of constant efficiency (c.f. paragraph 3.2).

Let us suppose that, at a certain date, in the sector devoted to the production of the commodity G, the bulk of current output is carried out by plants of the same age, and such age corresponds to the final part of their planned life n^*, in which, at the ruling price p_g, variable costs approximate revenues. At this time, the availability of a new and before unpredicted technology, after a period $\gamma<n$, becomes certain, and it is estimated that at r such technology is associated to the price p_g^f (where the apex f stands for future) inferior to p_g[30]. Provided γ is not very low, most of the plant under scrutiny will be renewed immediately, adopting the old technology again. This is because waiting until the initially planned n^* to renew the same plant would increase the period of overlapping between the old and the new "cheaper" technology. On the other hand, waiting until the date of possible introduction of the new technology would mean incurring losses (from the afterwards of the planned life to the adoption of the new technology). In this framework, three scenarios are open for the determination of the price p_g' that, during the time-span γ, guarantees: i) the total amortization of the plant; ii) profits at the rate r:

a) when the new technology is introduced, the corresponding price p_g^f, applied to the output of the plants aged γ, that embody the old technology, generates revenues greater than their operating costs. In this case the life of such plants will go on until the equality between revenues and current operating costs is reached. The solution relative to p_g' can still be found by basing the analysis on equation (19), but introducing the following modifications: i) the net revenues that accrue during the time-span the old and the new technology overlap, duly discounted for each unit of such period at the time of the implementation of initial investment I, have to be subtracted from the left-hand side of (19); ii) γ has to be substituted for n, *playing though the role of a known parameter and not of an unknown*; iii) the room for the capital cost, in the last operating period before the introduction of the new technology (i.e. period γ), is not zero, but it is $G_\gamma p_g' - V_n$, and as a consequence the term: $(G_\gamma p_g' - V_\gamma)[(1+r)^\gamma - 1]/(1+r)^\gamma$, has to be added in the right-hand side of (19); iv) the unknown p_g' must be substituted for the term V_n/G_n[31];
b) the price p_g^f, applied to the output of the plants aged γ, embodying the old technology, is equal or inferior to their current operating costs, that during the time-span γ are semi-increasing (and/or the output is semi-decreasing). The procedure is similar to the case a), except that the plants embodying the old

[30] The price p_g^f is determined in the "normal" way, if no further new technology is at the horizon. In the opposite case, its determination entails the iteration of the procedure we are going to describe.

[31] So the general expression corresponding to (19) is:

$$I - \frac{G_{\gamma+1}p_g^{\,f} - V_{\gamma+1}}{(1+r)^{\gamma+1}} - \frac{G_{\gamma+2}p_g^{\,f} - V_{\gamma+2}}{(1+r)^{\gamma+2}} - \dots - \frac{G_{\gamma+s}p_g^{\,f} - V_{\gamma+s}}{(1+r)^{\gamma+s}} = \left(G_\gamma p_g' - V_\gamma\right)\frac{(1+r)^\gamma - 1}{r(1+r)^\gamma} +$$

$$\left(\Delta V_{\gamma,\gamma-1} - \Delta G_{\gamma,\gamma-1}\, p_g'\right)\frac{(1+r)^{\gamma-1} - 1}{r(1+r)^{\gamma-1}} + \left(\Delta V_{\gamma-1,\gamma-2} - \Delta G_{\gamma,\gamma-1}\, p_g'\right)\frac{(1+r)^{\gamma-2} - 1}{r(1+r)^{\gamma-2}} + \dots + \left(\Delta V_{2,1} - \Delta G_{2,1}\, p_g'\right)\frac{(1+r) - 1}{r(1+r)}$$

where s is the last period in which the relevant numerator is greater than zero, and the only unknown is p_g'.

technology are dismissed when the new technology is introduced, and therefore negative terms on the left-hand side of (19) must not be introduced;

c) the same as above, but during the time-span γ no variation of current costs and/or of output occurs (i.e., constant efficiency rules), so that on the right-hand side of the relation in footnote 31 only the "new term" with respect to (19) – i.e. the one where the power γ appears – must be considered. The latter in turn can be written as: $(Gp_g'-V)[(1+r)^{\gamma}-1]/[(1+r)^{\gamma}]$, where G and V, given the constancy of the efficiency, stand respectively for a same level of physical output and for a same value of current variable costs, relative to each time-unit from 1 to γ. Therefore, the relation in footnote 31 may be written in the form: $Ir(1+r)^{\gamma} / [(1+r)^{\gamma}-1] + V = Gp_g'$, that corresponds exactly (c.f. footnote 7 − of course γ and n play the same role) to Sraffa's formula for the constant efficiency case with predetermined life for the plant!

Naturally, in order to make the above representation of technical progress nearer to reality, many developments have to be introduced. The most important are: a) it is necessary to consider the whole spectre of technologies in the future, in the hypothesis of perfect knowledge, in order to eliminate windfall gains or windfall losses; b) to allow in general, vice versa, imperfect knowledge, and to consider specific risk factors in the determination of the price as a consequence. The approach outlined above seems able to incorporate such developments. *In any case, unless technology changes only at time-spans exactly corresponding to the optimal life of the plant, or to its multiples, a determination of the price of the type described above is needed.*

4.2.2 *Plurality of capital goods.* Normally, a capital good, operating singularly with the application of specific labor and specific intermediate input, performs a definite operation, that constitutes a part of the chain of production of a commodity destined for the market. Marris (1964, p. 28) defines the performing of these separate operations as activities. A figurative price can be assigned to the "product" of each activity, starting from the ones carried out on the intermediate input that is purchased. At the end a "chain" of figurative values will emerge, that concur in determining the value of the commodities effectively sold on the market.

Equations (19) or (20) can clearly be used to correctly identify the figurative price of the product of each activity. In this way, although many plants are used for a single commodity for the market, each of them is amortized separately, and is connected to a specific "output price".

Let us now consider the case that two or more plants carry out jointly the same operation, i.e. they are part of a same activity (examples are: anvil and hammer; hardware and software of computers). For each capital good of this kind, it is possible to calculate the variation of current costs and/or of output by simulating the differences with respect to the hypothesis that the other jointly functioning "machine(s)" is (are) new. Eventual residuals due to joint ageing can be attributed to the various unities of the group according to "rules of thumb", as the recourse to the relative initial values. At this point the relations (19) or (20) – or the one in footnote 31, when there is technical progress – may be calculated for each unity of the couple (group). This allows us to determine the optimal life of each unity, and, in correspondence, its "contribution" to the optimal output price. Therefore, the procedure is similar to the multi-activity case.

4.2.3 *Variability of techniques*. Four analytical steps are considered.

A) The variability of the optimal life of the plant according to the rate of profit may be considered a first example of indirect change of techniques of production within a certain technological context (c.f. paragraph 4.1.4). Another source of variability, linked to fixed capital, concerns the possibility of different "strategies" of maintenance of its efficiency. Let us consider this possibility in the simplest case, i.e. with regard to equation (16), where the output is stable and only increases in variable costs are hypothesized. Suppose that two "strategies" are available. The first one corresponds to the exercise that has been carried out in par.3.5 (figure n.1; tables 1 and 2). The second one differs only due to three "deviations": expenses inferior by 2.5 regarding $\Delta V_{2,1}$; expenses superior by 4 regarding $\Delta V_{5,4}$; and again inferior by 1.6 regarding $\Delta V_{8,7}$. At the rate of profit zero (and for rates near zero) the first strategy is convenient (i.e. generates a lower n^*); at r=0,10 is the second strategy to produce a lower n^*; at r=0,20 the first strategy turns convenient again[32]. This example demonstrates that the reswitching of techniques is possible, even considering a single sector isolated from the rest of the economy. Of course, this result corresponds only to the point of view of the "strategies planner" in the sector producing the commodity G, who does not consider the possible changes of the relative costs of the various intermediate input, that occur, both for basic and non basic commodities, when r assumes different values. But these changes may compensate each other, as noted in paragraph 4.1.4. *Moreover, the reswitching surely becomes obtainable in a "single sector" economy, i.e. in a theoretical context where up to now this phenomenon has been excluded.*

B) Let us now consider the usual hypothesis that there is a link between output and variable costs, and the former may assume different levels. The simplest case is when output is set at different levels for each period, following an "external" rule (for example the maximum production corresponding to a single shift, or to the maximum possible utilization of the plant – all the available hours), and the link with the variable costs is of linear type, being simply expressed by a parameter (for example, the parameter o). So, we have: $V_1 = o_1 G_1$; $V_2 = o_2 G_2$; ... $V_{n-1} = o_{n-1} G_{n-1}$; $V_n = o_n G_n$; where, in the general case of decreasing efficiency (c.f. paragraph 4.1.2), the inequalities $G_n \leq G_{n-1} \leq \ldots \leq G_2 \leq G_1$ and $o_n \geq o_{n-1} \geq \ldots \geq o_2 \geq o_1$ hold. The i^{th} multiplicand of the right-hand side of the mirror equation (19), i.e. the sum

[32] In these hypotheses the right-hand side of the mirror equation relative to the two strategies differs simply by: $-2{,}5/(1+r) + 4/(1+r)^4 - 1{,}6/(1+r)^7$: the factor in r being different from (16), since the trend of variation of the variable costs is not changed by the existence of "deviations". For the same reason, when the result of the above difference is negative (as it happens for r=0 and for r=0,2)), in order to satisfy (16), n^* (and as a consequence the price) must be greater for the second strategy. If, on the contrary, such a result is positive (as for r=0,1), n^* and the price are greater for the first strategy.

$\Delta V_{i,i-1}$+ (-ΔG_i) V_n/G_n , where V_n/G_n , according to (16), corresponds to the price p_g, is expressed by: $o_iG_i - o_{i-1}G_{i-1} + p_g (G_i - G_{i-1}) = G_i (p_g - o_i) - G_{i-1} (p_g - o_{i-1})$. Each of the two terms of this last expression can be interpreted as the sides of a rectangle, one side representing the quantity and the other side representing gross unit margins, that, multiplied between them, generate the total gross margin available for amortization and interest-profit in periods *i* and *i*-1. The area of the various rectangles tends to decrease as *i* increases (since $G_i \leq G_{i-1}$ and $o_i \geq o_{i-1}$), reaching zero when $i=n$ (since in correspondence it is $p_g = o_nG_n/G_n = o_n$, and $o_n - o_n$ is zero). The optimal n^* is obtained when the summatory of the differences between successive areas becomes equal to *I*, being each difference multiplied by its own factor, based on the appropriate power of *r* according to (19).

The level of output can also be determined endogenously, in the traditional way, at the intersection between the marginal variable cost (*mc*) and the price p_g, by calculating the total gross margin as a rectangle which sides are respectively: 1) the "equilibrium" output, determined as just indicated; 2) the difference between the price p_g and the average variable cost (*ac*) correspondent to the equilibrium output[33]. In this case, the right-hand side of (19) may be represented in *Figure* 2, in which the last and the first appropriate multiplying factors are "linked" to the area to which they apply (respectively ABCD and EFGHIL).

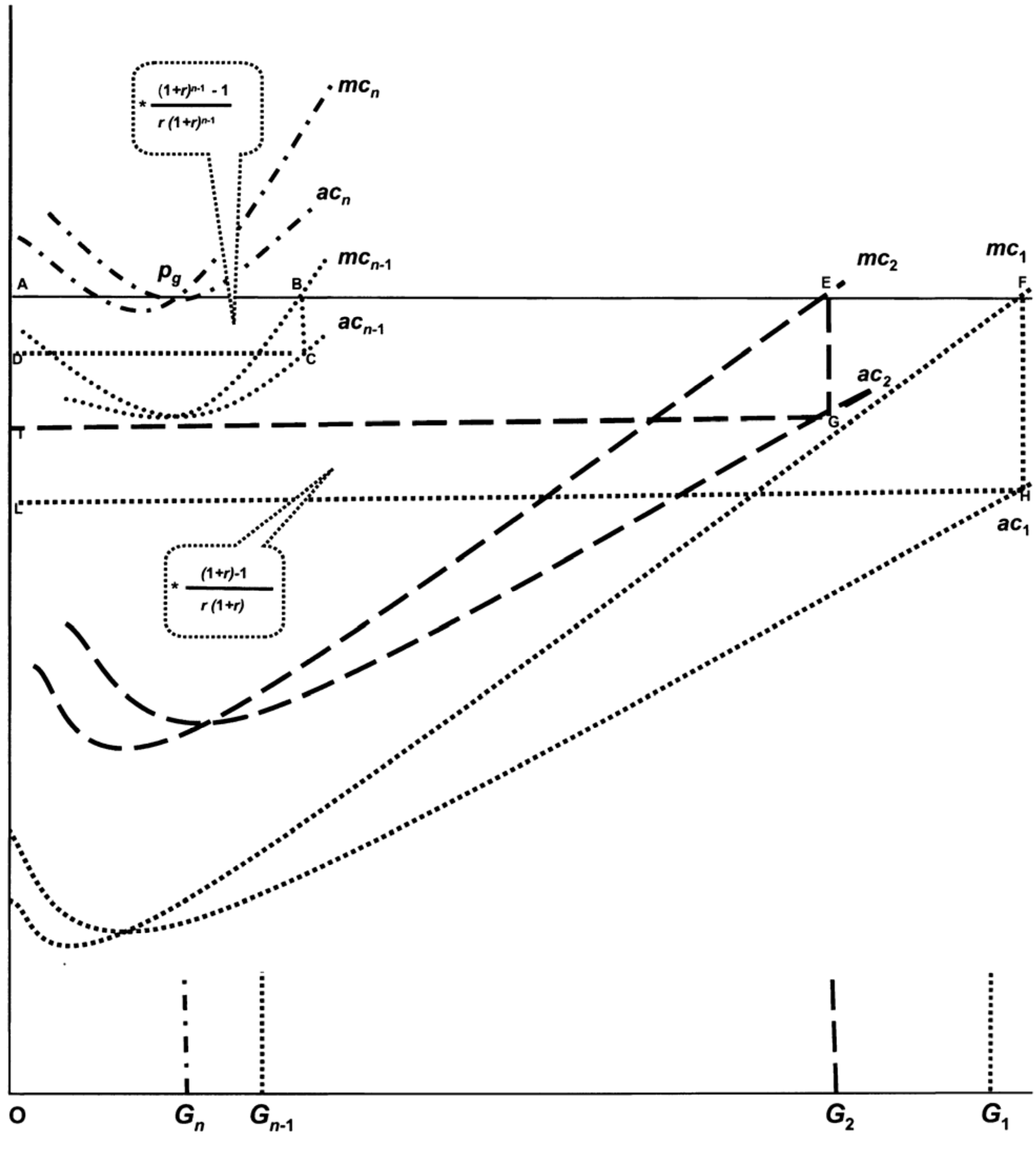
C) In *Figure* 2 it is implicitly hypothesized that the placement of the curves *mc* and *ac* relative to each period does not depend on the output of previous periods. In the opposite hypothesis, we have to consider "a bundle" of pairs *mc*-*ac* for each period except the first. Should the plant work only for the first period, the equilibrium price and quantity would be obtained at the intersection between mc_1 and ac_1. Going on, to "try" the second period as n^*, we can start positing p_g at the intersection of the "lowest" pair of the bundle mc_2 - ac_2, and setting G_2 at the correspondent level in the *x* axis. As a consequence, G_1 will rise with respect to the case of only one year functioning of the plant, the new level being determined by the intersection between mc_1 and the price correspondent to the lowest pair of the bundle mc_2 - ac_2. However, the "new" output G_1 will correspond to a pair of the bundle mc_2 - ac_2 situated above the lowest one. Consequently, a higher p_g than the first trial will result for the second period. The process must be carried on, up to the point where the interaction between G_2 and G_1 involves minimal changes, or G_1 reaches a ceiling (maximum physical utilization of the plant, institutional constraints determining a ceiling to utilization).

By following the same procedure, it is then possible to pass to period 3, and to any further period up to n^*, in which full amortization is obtained. In each passage the interaction must regard though not only the period "under trial" and the previous one, but all the previous periods back to the first one. In the end, a graph similar to that

[33]The difference of the equilibrium output among different periods may be due to a more intensive use of the plant in each moment of its utilization (more labor and more intermediate input), and/or to a longer "exploitation" of single parts of the time-span in which the period is divided (more overtime, more shifts, weekend shifts, and so on). The former aspect is the one ordinarily considered in the economic theory. The latter aspect has been investigated in depth by Marris (1964).

Figure 2

The equilibrium for the case of variability of techniques



of *Figure* 2 will emerge, but as the result of a process of selection of the average and marginal cost curves referred to each period.

D) In point A) we have considered different strategies to influence n^*, that produce a change of the profile of amortization and interest on capital, i.e. of fixed costs. In points B) and C) the possible different productive combinations have regarded variable costs in relation to output. As a matter of fact, in our analysis different strategies aimed at the maintenance of fixed capital involve changes of the dynamic of variable costs related to output, and different combinations between variable costs output imply different profiles of fixed costs (amortization and, as a consequence, interest on capital): *the traditional distinction between fixed and variable costs becomes deeply blurred*. The relation between level of output and amount of initial investment completes this basic overview on techniques. Such a relation may also be studied in *Figure* 2, by observing the consequences of the increase in the distance of the whole set of curves from the *y* axis, that expresses greater levels of output in relation to the "quantity" of plants. Let us assume that, due to commercial reasons, firms need "to stay in the market" with a stable quantity of output for sale, and therefore, in the final stage of a multi-activity production, in principle the best "capital asset" is constituted by plants of different ages (this happens because, as *Figure* 2 shows, the optimal output of the same plant may be very different depending on its age). However, there are normally limits to the divisibility of plants; and/or smaller plants, even operating at best, may be associated to greater unit total production costs.

If this is the case, a first consequence is that, to produce the given level of output, may be preferable to operate with fewer plants than in the case of perfect divisibility, and possibly with the same plant, which becomes progressively older. A second consequence is that, in this case, since in the initial periods of activity the output is less than the potential optimal one, and so a room for amortization and interest on capital inferior to the potential one is created, the life of capital and the price p_g are higher than in the case of perfect divisibility. As a third consequence, greater levels of output tend to reduce the problems brought about by non-perfect or limited divisibility, so that the output price tends to be lower at any given *r*. Of course this situation is amplified by the existence of many activities that lead up to the final stage of production, since also for the concerned plants the optimal output tends to differ according to age. In their case problems would remain even in the hypothesis that full variability of final output according to optimal utilization were to occur in the final stage, since it is unlikely that the distribution of the optimal output in any stage previous to the last one follows the same temporal pattern as the latter.

The overall conclusion is that the relation between quantity and unit total cost shows a decreasing profile, which may be intense and prolonged, due to the possibility to utilize capital in more and more efficient ways[34]. If the above relation

[34] The concept of declining unit total costs is preferable to that of increasing returns of capital, since capital, due to the changing of its composition and its operating modes as production increases, cannot be defined homogeneously in physical terms. Moreover, there is a continuous correlation between fixed and variable costs, since the optimal life of the various plants tend to change in a complicated way.

becomes stable at a certain point, *we enter a regime of constant total unit costs* (briefly: constant costs)*, which is the only other possible regime beyond the one just described* (briefly: decreasing costs). In fact, should costs increase above a certain level of output, the firms that operate at smaller quantities benefit from a competitive advantage.

4.2.4 *Implications.* Some brief considerations emerge, in particular as regards the marginalist theory:

a) the marginalist array of average-marginal cost curves at a given level of "capital", integrated in Sraffa's framework, as done in the previous analysis, constitutes not *the* supply side of the model for the determination of prices, but *a part* of the supply side. *Moreover, the addition of this array does not destroy, but rather improves the capability of Sraffa's approach to determine "supply-side" prices*;

b) vice versa, there is no way of consistently integrating in the model the concept of marginal product of capital, and, as a consequence, of establishing any meaningful relation between this concept and the rate of profit. This is because the analysis developed above does not imply a change of the model of the determination of the price[35]: *specifically, there is no necessity for an endogenous determination of r.* This is not a problem, though, on the contrary, a double advantage emerges. First: it is possible to maintain a "degree of freedom" for the determination of *r*. Second: the determination of prices in a regime of decreasing costs, also in a general equilibrium context, may be explored more easily (c.f. for both aspects the final part of this work);

c) the particular combination between Sraffa's and the marginalist's approaches outlined in this work shows a great potential to further illuminate many fields of applied economics. One needs only to consider those that have been noted: a proper calculation of amortization; a proper determination of the cost of the product (the basis for a correct formation of the price bid by firms), when many activities are involved; a correct analysis of the optimal utilization of plants at any age. The application regarding the macroeconomic problems and policies is even more promising, as demonstrated below.

5. Possible further developments

The previous analysis on prices certainly maintains the basic characteristics of the classical approach. These are that the output is given, and the solutions are found after fixing one of the two distributive variables: real wage or (as in our case) the rate of profit. The only difference is that variability of techniques has been explicitly

[35] With reference to *Figure* 2, in the case of constant costs nothing changes, except that the whole array of curves moves towards the right at the same pace as output increases. In the case of decreasing costs, the move of the curves towards the right is accompanied by a change of their relative position, and normally by a lower equilibrium n^*, so that the maintenance of total amortization of plants at the given *r* is compatible with a decrease of p_g, and/or with the creation of rents (c.f. paragraph 5).

introduced, starting from the particular case of the choice among different determinations of capital life (which can well be classified as a techniques choice, since decisions on the use of capital are involved). Now let us consider briefly two possible major developments: a) the determination of "total supply"; b) the determination of *r*.

As regards the first problem, the way is open for Leijonhufvud's approach, based on big differences for the economic system according to the regime of costs which is prevailing (Leijonhufvud, 1995). If constant costs are dominant, and they also regard low levels of output (c.f. paragraph 4.2.3, point D), the model "labor hiring capital" can be adopted. This means that total supply tends naturally to full employment (apart from Keynesian problems), since, in the contrary case, it is easy to enter the "entrepreneurial field". Specifically, entrance in such a field may be pursued by unemployed labor, or by workers who could be attracted by eventual entrepreneurial rents, fostered by sticky prices and falling wages. On the contrary, if decreasing costs prevail, or constant costs may be reached only at very high levels of production, the difficulty to enter the "entrepreneurial field" brings about the dominance of the model "capital hiring labor". The implication is that the tendency to structural full employment is no longer guaranteed. Moreover, as regards the distribution of income, ample room is disclosed for what Leijonhufvud calls "joint rents" in the decreasing costs sectors (i.e. profits, and wages as well, that are significantly greater than the ones prevailing in the constant returns sectors)[36].

On the second problem, the way is open for Sraffa's indication of taking as a standing point the rate of interest of the monetary market. More precisely, it is opportune to start from an analysis of the forces that determine the "structural level" of investment and saving in the economy (by way of example, technical progress and increase of population for the former; thrift and characteristics of the welfare state for the latter), so generating the "long term" rate of interest to which the rate of profit prevailing in the economic activity is linked. Then, the determination of the rate of interest due to short term fluctuations of the two magnitudes may be examined, with their consequences in the short term equilibrium of financial markets. So "Keynesian problems" of the economy can be treated along the lines of the relation between short term and long term expectations regarding the rate of interest, as Keynes himself basically wished.

These two developments of the basic analysis of this work have connections between them, but may be elaborated separately from each other, and separately from the treatment of the determination of prices. Another fundamental feature of classical economics, i.e. a theoretical structure characterized by "separate" logical stages, is so maintained (c.f. Garegnani, 1989).

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[36] When joint rents are considered, the determination of p_g changes with respect to paragraph 4.2.4, since the unit amount of such rents may enter the calculations. Note that, as regards the part of rents which produces an increase of gross margins, the comparison among different productive sectors does not need to adopt the value amount of capital as a standpoint. Added value, and the value of production (especially the latter), may in fact constitute a better standard.
Another specific aspect of the price formation with decreasing costs is that, beyond the total output of the sector, also the number of firms and the distribution of output among them must be indicated.

References

Baldone, S. (1980), *Fixed Capital in Sraffa's Theoretical Scheme,* in: L. Pasinetti (Ed.) *Essays on the Theory of Joint Production* (London and Basingstoke, MacMillan).
Garegnani, P. (1992), *Sraffa: Classical versus Marginalist Analysis*, in: K. Bharadway & B. Schefold (Eds.) *Essays on Piero Sraffa* (London, Routledge).
Kurtz, H.D. & Salvadori, N. (1995), *Theory of Production. A Long Period Analysis* (Cambrige, Cambridge University Press).
Leijonhufvud, A. (1995), *The Individual, the Market and the Industrial Division of Labor,* in: C. Mongardini (Ed.) *L'individuo e il mercato* (Roma, Bulzoni Editore).
Marris, R. (1964), *The Economics of Capital Utilisation* (Cambridge, Cambridge University Press).
Roncaglia, A. (1971), Il capitale fisso in un modello di accumulazione circolare, *Studi Economici*, pp. 1-14.
Schefold, B. (1980), *Fixed Capital as a Joint Product and the Analysis of Accumulation with Different Forms od Technical Progress,* in: L. Pasinetti (Ed.) *Essays on the Theory of Joint Production* (London and Basingstake, MacMillan).
Sraffa, P. (1960), *Production of Commodities by means of Commodities* (Cambridge, Cambridge University Press).